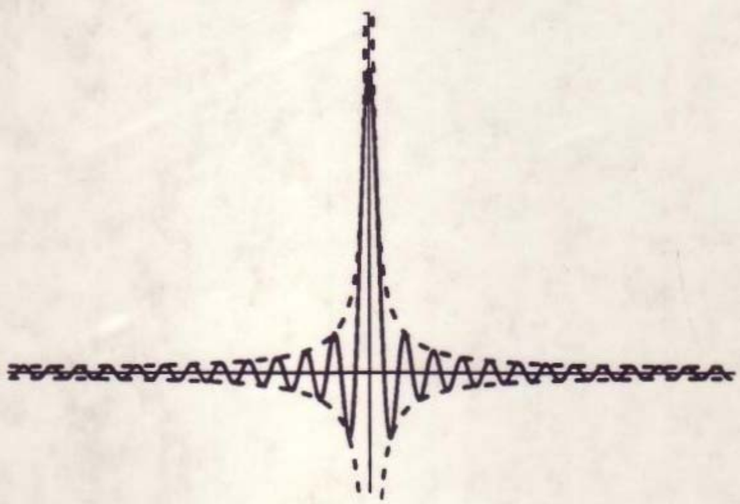
FOURIER ANALYSIS AND ITS APPLICATIONS

Gerald B. Folland



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PREFACE

This book is intended for students of mathematics, physics, and engineering at the advanced undergraduate level or beyond. It is primarily a text for a course at the advanced undergraduate level, but I hope it will also be useful as a reference for people who have taken such a course and continue to use Fourier analysis in their later work. The reader is presumed to have (i) a solid background in calculus of one and several variables, (ii) knowledge of the elementary theory of linear ordinary differential equations (i.e., how to solve first-order linear equations and second-order ones with constant coefficients), and (iii) an acquaintance with the complex number system and the complex exponential function $e^{x+iy} = e^x(\cos y + i \sin y)$. In addition, the theory of analytic functions (power series, contour integrals, etc.) is used to a slight extent in Chapters 5, 6, 7, and 9 and in a serious way in Sections 8.2, 8.4, 8.6, 10.3, and 10.4. I have written the book so that lack of knowledge of complex analysis is not a serious impediment; at the same time, for the benefit of those who do know the subject, it would be a shame not to use it when it arises naturally. (In particular, the Laplace transform without analytic functions is like Popeye without his spinach.) At any rate, the facts from complex analysis that are used here are summarized in Appendix 2.

The subject of this book is the whole circle of ideas that includes Fourier series, Fourier and Laplace transforms, and eigenfunction expansions for differential operators. I have tried to steer a middle course between the mathematics-for-engineers type of book, in which Fourier methods are treated merely as a tool for solving applied problems, and the advanced theoretical treatments aimed at pure mathematicians. Since I thereby hope to please both the pure and the applied factions but run the risk of pleasing neither, I should give some explanation of what I am trying to do and why I am trying to do it.

First, this book deals almost exclusively with those aspects of Fourier analysis that are useful in physics and engineering rather than those of interest only in pure mathematics. On the other hand, it is a book on *applicable* mathematics rather than *applied* mathematics: the principal role of the physical applications herein is to illustrate and illuminate the mathematics, not the other way around. I have refrained from including many applications whose principal conceptual content comes from Subject X rather than Fourier analysis, or whose appreciation requires specialized knowledge from Subject X; such things belong more properly in a book on Subject X where the background can be more fully explained. (Many of my favorite applications come from quantum physics, but in accordance with this principle I have mentioned them only briefly.) Similarly, I have not worried too much about the physical details of the applications studied here. For example, when I think about the 1-dimensional heat equation I usually envision a long thin rod, but one who prefers to envision a 3-dimensional slab whose temperature varies only along one axis is free to do so; the mathematics is the same.

Second, there is the question of how much emphasis to lay on the theoretical aspects of the subject as opposed to problem-solving techniques. I firmly believe that theory — meaning the study of the ideas underlying the subject and the reasoning behind the techniques — is of intellectual value to everyone, applied or pure. On the other hand, I do not take "theory" to be synonymous with "logical rigor." I have presented complete proofs of the theorems when it is not too onerous to do so, but I often merely sketch the technical parts of an argument. (If the technicalities cannot easily be filled in by someone who is conversant with such things, I usually give a reference to a complete proof elsewhere.) Of course, where to draw the line is a matter of judgment, and I suppose nobody will be wholly satisfied with my choices. But those instructors who wish to include more details in their lectures are free to do so, and readers who tire of a formal argument have only to skip to the end-of-proof sign ▮. Thus, the book should be fairly flexible with regard to the level of rigor its users wish to adopt.

One feature of the theoretical aspect of this book deserves special mention. The development of Lebesgue integration and functional analysis in the period 1900–1950 has led to enormous advances in our understanding of the concepts underlying Fourier analysis. For example, the completeness of L^2 and the shift from pointwise convergence to norm convergence or weak convergence simplifies much of the discussion of orthonormal bases and the validity of series expansions. These advances have usually not found their way into application-oriented books because a rigorous development of them necessitates the building of too much machinery. However, most of this machinery can be ignored if one is willing to take a few things on faith, as one takes the intermediate value theorem on faith in freshman calculus. Accordingly, in §3.3–4 I assert the existence of an improved theory of integration, the Lebesgue integral, in the context of which one has (i) the completeness of L^2, (ii) the fact that "nice" functions are dense in L^2, and (iii) the dominated convergence theorem. I then proceed to use these facts without further ado. (The dominated convergence theorem, it should be noted, is a wonderful tool even in the context of Riemann integrable functions.) Later, in Chapter 9, I develop the theory of distributions as linear functionals on test functions, the motivation being that the value of a distribution on a test function is a smeared-out version of the value of a function at a point. Discussion of functional-analytic technicalities (which are largely irrelevant at the elementary level) is reduced to a minimum.

With the exception of the prerequisites and the facts about Lebesgue integration mentioned above, this book is more or less logically self-contained. However, certain assertions made early in the book are established only much later:

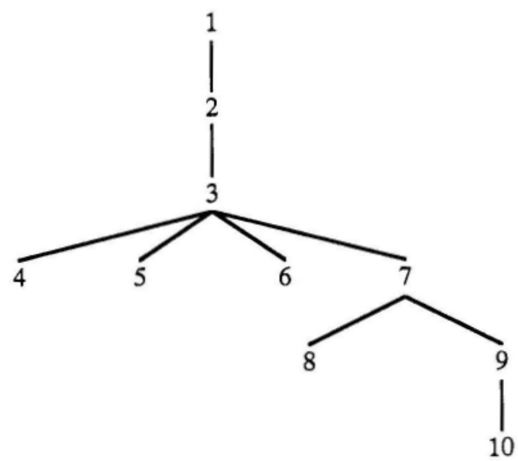
(i) The completeness of the eigenfunctions of regular Sturm-Liouville problems is stated in §3.5 and proved, in the case of separated boundary conditions, in §10.3.

(ii) The asymptotic formulas for Bessel functions given in §5.3 are proved via Watson's lemma in §8.6.

(iii) The proofs of completeness of Legendre, Hermite, and Laguerre polynomials in Chapter 6 rely on the Weierstrass approximation theorem and the Fourier inversion theorem, proved in Chapter 7.

(iv) The discussion of weak solutions of differential equations in §9.5 justifies many of the formal calculations with infinite series in the earlier chapters.

Thus, among the applications of the material in the later part of the book is the completion of the theory developed in the earlier part.



CHAPTER DEPENDENCE DIAGRAM

The main dependences among the chapters are indicated in the accompanying diagram, but a couple of additional comments are in order.

First, there are some minor dependences that are not shown in the diagram. For example, a few paragraphs of text and a few exercises in Sections 6.3, 7.5, 8.1, and 8.6 presuppose a knowledge of Bessel functions, but one can simply omit these bits if one has not covered Chapter 5. Also, the discussion of techniques in §4.1 is relevant to the applied problems in later chapters, particularly in §5.5.

Second, although Chapter 10 depends on Chapter 9, except in §10.2 the only part of distribution theory needed in Chapter 10 is an appreciation of delta functions on the real line and the way they arise in derivatives of functions with jump discontinuities. Hence, one could cover Sections 10.1 and 10.3–4 after an informal discussion of the delta function, without going through Chapter 9.

There is enough material in this book for a full-year course, but one can also select various subsets of it to make shorter courses. For a one-term course one could cover Chapters 1–3 and then select topics *ad libitum* from Chapters 4–7. (If one wishes to present some applications of Bessel functions without discussing the theory in detail, one could skip from the recurrence formulas in §5.2 to the statement of Theorem 5.3 at the end of §5.4 without much loss of continuity.) I have taught a one-quarter (ten-week) course from Chapters 1–5 and a sequel to it from Chapters 7–10, omitting a few items here and there.

One further point that instructors should keep in mind is the following. Most of the book deals with rather concrete ideas and techniques, but there are two places where concepts of a more general and abstract nature are discussed in a serious way: Chapter 3 (L^2 spaces, orthogonal bases, Sturm-Liouville problems) and Chapter 9 (functions as linear functionals, generalized functions). These parts are likely to be difficult for students who have had little experience with abstract mathematics, and instructors should plan their courses accordingly.

Fourier analysis and its allied subjects comprise an enormous amount of mathematics, about which there is much more to be said than is included in this book. I hope that my readers will find this fact exciting rather than dismaying. Accordingly, I have included a sizable although not exhaustive bibliography of books and papers to which the reader can refer for more information on things that are touched on lightly here. Most of these references should be reasonably accessible to the students for whom this book is primarily intended, but a few of them are of a considerably more advanced nature. This is inevitable; the topics in this book impinge on a lot of sophisticated material, and the full story on some of the things discussed here (singular Sturm-Liouville problems, for instance) cannot be told without going to a deeper level. But these advanced references should be of use to those who have the necessary background, and may at least serve as signposts to those who have yet to acquire it.

I am grateful to my colleagues Donald Marshall, Douglas Lind, Richard Bass, and James Morrow and to the students in our classes for pointing out many mistakes in the first draft of this book and suggesting a number of improvements. I also wish to thank the following reviewers for their helpful suggestions in revising the manuscript: Giles Auchmuty, University of Houston; James Herod, Georgia Institute of Technology; Raymond Johnson, University of Maryland; Francis Narcowich, Texas A & M University; Juan Carlos Redondo, University of Michigan; Jeffrey Rauch, University of Michigan; Jesus Rodriguez, North Carolina State University; and Michael Vogelius, Rutgers University.

Gerald B. Folland

CONTENTS

CHAPTER 1
OVERTURE

The subject of this book is *Fourier analysis,* which may be described as a collection of related techniques for resolving general functions into sums or integrals of simple functions or functions with certain special properties. Fourier analysis is a powerful tool for many problems, and especially for solving various differential equations of interest in science and engineering. The purpose of this introductory chapter is to provide some background concerning partial differential equations. Specifically, we introduce some of the basic equations of mathematical physics that will provide examples and motivation throughout the book, and we discuss a technique for solving them that leads directly to problems in Fourier analysis.

At the outset, let us present some notations that will be used repeatedly. The real and complex number systems will be denoted by **R** and **C**, respectively. We shall be working with functions of one or several real variables $x_1, \dots, x_n$. We shall denote the ordered n-tuple $(x_1, \dots, x_n)$ by $\mathbf{x}$ and the space of all such ordered n-tuples by $\mathbf{R}^n$.

In most of the applications, n will be 1, 2, 3, or 4, and the variables x_j will denote coordinates in one, two, or three space dimensions, together with time. In this situation we shall usually write x, y, z instead of x_1, x_2, x_3 for the spatial variables, and we shall denote the time variable by t. Moreover, we shall use the common subscript notation for partial derivatives:

$$u_x = \frac{\partial u}{\partial x}, \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2} \qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \qquad \text{etc.}$$

A function f of one real variable is said to be of class $C^{(k)}$ on an interval I if its derivatives $f', \dots, f^{(k)}$ exist and are continuous on I. Similarly, a function of n real variables is said to be of class $C^{(k)}$ on a set $D \subset \mathbf{R}^n$ if all of its partial derivatives of order $\leq k$ exist and are continuous on D. If the function possesses continuous derivatives of all orders, it is said to be of class $C^{(\infty)}$.

Finally, we use the common notation with square and round brackets for closed and open intervals in the real line **R**:

$$[a, b] = \{x : a \leq x \leq b\}, \qquad (a, b) = \{x : a < x < b\},$$
$$[a, b) = \{x : a \leq x < b\}, \qquad (a, b] = \{x : a < x \leq b\}.$$

1.1 Some equations of mathematical physics

In order to understand the significance of the ideas as they arise, it will be useful to have a few physical applications in mind as examples of the sort of problems we are trying to solve. Accordingly, we begin with a brief and informal discussion of some of the basic partial differential equations of classical mathematical physics. These equations all involve a fundamental differential operator known as the Laplacian, which is defined as follows. If u is a function of the real variables $x_1, \ldots, x_n$ of class $C^{(2)}$, the **Laplacian** of u is the function $\nabla^2 u$ defined by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \tag{1.1}$$

The first of the equations we shall study is the **wave equation**:

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \tag{1.2}$$

Here u represents a wave traveling through an n-dimensional medium—where, in practice, n will usually be 1, 2, or 3. More precisely, $x_1, \ldots, x_n$ are the coordinates of a point $\mathbf{x}$ in the medium; t is the time; c is the speed of propagation of waves in the medium; and $u(\mathbf{x}, t)$ is the amplitude of the wave at position $\mathbf{x}$ and time t.

The wave equation provides a reasonable mathematical model for a number of physical processes, such as the following:

(a) Vibrations of a stretched string, such as a guitar string.
(b) Vibrations of a column of air, such as an organ pipe or clarinet.
(c) Vibrations of a stretched membrane, such as a drumhead.
(d) Waves in an incompressible fluid, such as water.
(e) Sound waves in air or other elastic media.
(f) Electromagnetic waves, such as light waves and radio waves.

The number n of spatial dimensions is 1 in examples (a) and (b), 2 in examples (c) and (d) (since the waves appear on the *surface* of the water), and 3 in examples (e) and (f). In (a), (c), and (d), u represents the transverse displacement of the string, membrane, or fluid surface; in (b) and (e), u represents the longitudinal displacement of the air; and in (f), u is any of the components of the electromagnetic field.

We shall not attempt to derive the wave equation from physical principles here, since each of the preceding examples involves different physics. Examples (a) and (f) are explained in Appendix 1; discussions of the others may be found, for example, in Ingard [32]* and Taylor [51]. We should point out, however, that in most cases the derivation involves making some simplifying assumptions. Hence, the wave equation gives only an approximate description of the actual physical process, and the validity of the approximation will depend on whether

* Numbers in brackets refer to the bibliography at the end of the book.

certain physical conditions are satisfied. For instance, in example (a) the vibrations should be small enough so that the string is not stretched beyond its limits of elasticity. In example (f) it follows from Maxwell's equations, the fundamental equations of electromagnetism, that the wave equation is satisfied *exactly* in regions containing no electric charges or currents — but of course the assumption of no charges or currents can only be approximately valid in the real world. (Of course, it is precisely the fact that the wave equation is only an approximation that allows it to be a useful model in so many different situations!)

The next basic differential equation on our list is the **heat equation**:

$$u_t = k\nabla^2 u. \tag{1.3}$$

This equation describes the diffusion of thermal energy in a homogeneous material (that is, one whose composition does not change from point to point). As in the wave equation, the variables x_j are spatial coordinates and t is time, but now $u(\mathbf{x}, t)$ is the *temperature* at a position $\mathbf{x}$ and time t, and k is a constant called the "thermal diffusivity" of the material. A brief derivation is given in Appendix 1. As for the number of spatial variables, the case $n = 3$ is the most fundamental from the physical point of view, but the cases $n = 1$ and $n = 2$ are also of interest as models of situations where the heat flow is practically all in one or two directions. For example, the heat equation with $n = 1$ can be used to describe heat flow along a wire or rod, provided that heat flow in directions perpendicular to the axis of the rod can be neglected. It can also be used to describe heat flow in a slab of material, such as a wall separating two rooms, where only the heat flow from one room toward the other (as opposed to flow in directions parallel to the wall) is significant.

Two warnings: (i) The heat equation can be used to model heat flow in both solids and fluids (liquids and gases), but in the latter case it does *not* take any account of the phenomenon of convection; that is, it will provide a reasonable model only if conditions are such as to exclude any macroscopic currents in the fluid. (ii) The heat equation is not a fundamental law of physics, and it does not give reliable answers at very low or very high temperatures. In particular, it is obvious that if u is a solution then so is $u + c$ for any constant c; thus the heat equation does not recognize the existence of absolute zero!

The heat equation can also be used to model other diffusion processes. For example, if a drop of red dye is placed in a body of water, the dye will gradually spread out and permeate the entire body. If convection effects are negligible, equation (1.3) will describe the diffusion of the dye through the water ($u(\mathbf{x}, t)$ now being the concentration of dye at position $\mathbf{x}$ and time t).

Next, we come to the **Laplace equation**:

$$\nabla^2 u = 0. \tag{1.4}$$

Laplace's equation arises in a number of different contexts. It is satisfied by the electrostatic potential in any region containing no electric charge, and by the gravitational potential in any region containing no mass. It is also the equation that governs standing waves and steady-state heat distributions — that is, solutions of the wave equation and the heat equation that are independent of time. We shall meet other applications of it later on.

Partial differential equations such as the ones discussed above typically have solutions in such great abundance that there is no reasonable way of giving an explicit description of all of them. The most common way of pinning down a particular solution is to impose some boundary conditions. Different types of differential equations require different types of boundary conditions, and the particular conditions that are appropriate for a given physical problem will depend on the particular physical situation. The physics is generally a good guide to the mathematics: "reasonable" physical conditions usually lead to "reasonable" mathematical problems.

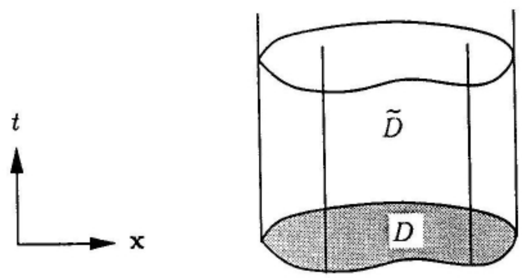


FIGURE 1.1. The region D in $\mathbf{x}$-space and the region $\widetilde{D}$ in $\mathbf{x}t$-space.

These matters may best be explained by examining a few examples. Let us consider the heat equation: suppose we are interested in studying the diffusion of heat in a body that occupies a bounded region D of $\mathbf{x}$-space, given the initial temperature distribution in the body. That is, we wish to solve the heat equation (1.3) in the region

$$\widetilde{D} = \left\{(\mathbf{x}, t) : \mathbf{x} \in D, \quad t > 0\right\}$$

of $(\mathbf{x}, t)$-space subject to the initial condition

$$u(\mathbf{x}, 0) = f(\mathbf{x}), \tag{1.5}$$

where $f(\mathbf{x})$ is the temperature distribution at time $t = 0$. (See Figure 1.1.) Equation (1.5) is a condition on u on the "horizontal" part of the boundary of $\widetilde{D}$, but it is not enough to specify u completely; we also need a boundary condition on the "vertical" part of the boundary to tell what happens to the heat when it reaches the boundary surface S of the spatial region D. Here the particular physical conditions at hand must be our guide. One reasonable assumption is that S is held at a constant temperature u_0 (for example, by immersing the body in a bath of ice water), thus:

$$u(\mathbf{x}, t) = u_0 \quad \text{for } \mathbf{x} \in S, \ t > 0. \tag{1.6}$$

Another reasonable assumption is that D is insulated, so that no heat can flow in or out across S. Mathematically, this amounts to requiring the *normal derivative* of u along the boundary S to vanish:

$$(\nabla u \cdot \mathbf{n})(\mathbf{x}, t) = 0 \quad \text{for } \mathbf{x} \in S,\ t > 0. \tag{1.7}$$

Here $\mathbf{n}$ is the unit outward normal vector to S (and we are implicitly assuming that the surface S is smooth, so that $\mathbf{n}$ is well-defined). A more realistic assumption than either (1.6) or (1.7) is that the region outside D is held at a constant temperature u_0, and the rate of heat flow across the boundary S is proportional to the difference in temperatures on the two sides:

$$(\nabla u \cdot \mathbf{n})(\mathbf{x}, t) + a\Big(u(\mathbf{x}, t) - u_0\Big) = 0 \quad \text{for } \mathbf{x} \in S,\ t > 0. \tag{1.8}$$

This is **Newton's law of cooling,** and $a > 0$ is the proportionality constant. The conditions (1.6) and (1.7) may be regarded as the limiting cases of (1.8) as $a \to \infty$ or $a \to 0$.

At any rate, it turns out that the initial condition (1.5) together with any one of the boundary conditions (1.6), (1.7), or (1.8) leads to a *well-posed* problem: one having a unique solution that depends continuously (in some appropriate sense) on the initial data f. The same discussion is also valid for the heat equation in one or two space dimensions. (In one space dimension, the "region" D is just an interval in the x-axis, and the "normal derivative" $\nabla u \cdot \mathbf{n}$ is just u_x at the right endpoint and $-u_x$ at the left endpoint.)

A similar analysis applies to boundary value problems for the wave equation (1.2), with one significant difference: the wave equation is second-order in the time variable t, whereas the heat equation is only first-order in t. For this reason, in solving the wave equation it is appropriate to specify not only the initial values of u as in (1.5) but also the initial velocity u_t:

$$u(\mathbf{x}, 0) = f(\mathbf{x}), \qquad u_t(\mathbf{x}, 0) = g(\mathbf{x}) \quad \text{for } \mathbf{x} \in D. \tag{1.9}$$

The imposition of the initial conditions (1.9) together with a boundary condition of the form (1.6), (1.7), or (1.8) leads to a unique solution of the wave equation. For example, to analyze the motion of a vibrating string of length l that is fixed at both endpoints, we take the "region" D to be the interval $[0, l]$ on the x-axis and solve the one-dimensional wave equation with boundary conditions (1.6) (where $u_0 = 0$) and (1.9):

$$u_{tt} = c^2 u_{xx}, \qquad \begin{array}{l} u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x) \quad \text{for } 0 < x < l, \\ u(0,t) = u(l,t) = 0 \quad \text{for } t > 0. \end{array}$$

Remark: The "velocity" u_t is not the same as the constant c in the wave equation. c is the speed of propagation of the wave along the string, whereas u_t is the rate of change of the displacement of a particular point on the string. (The same is true for waves in media other than strings.)

The Laplace equation (1.4) is of a rather different character, as it does not involve time. The most important boundary value problem for this equation, the so-called **Dirichlet problem**, consists in specifying the values of u on the boundary of the region in question. That is, we solve $\nabla^2 u = 0$ in a region D subject to the condition that u agrees with a given function f on the boundary S of D. This is a well-posed problem when D is bounded and S is smooth (except perhaps for corners and edges). Another useful boundary value problem for Laplace's equation is the **Neumann problem**, which consists of specifying the values of the normal derivative $\nabla u \cdot \mathbf{n}$ on S:

$$\nabla^2 u = 0 \text{ in } D, \qquad (\nabla u \cdot \mathbf{n})(\mathbf{x}) = g(\mathbf{x}) \quad \text{for } \mathbf{x} \in S.$$

Here we do not quite have uniqueness, for if u is a solution, then so is $u + C$ for any constant C. Moreover, the boundary data g must satisfy the condition $\iint_S g = 0$ in order for a solution to exist, because by the divergence theorem,

$$\iint_S (\nabla u \cdot \mathbf{n})\, dS = \iiint_D \nabla^2 u\, dV = 0$$

for any u such that $\nabla^2 u = 0$. However, there are no other obstructions to existence and uniqueness; and since there is only one constant to be specified to obtain uniqueness, and only one linear equation to be satisfied to obtain existence, the Neumann problem is still regarded as well behaved.

There is one more point that should be mentioned in connection with the interpretation of boundary conditions. Suppose, for example, that we are interested in the initial value problem for the heat equation:

$$u_t = k\nabla^2 u \quad \text{for } t > 0, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}).$$

If one interprets this absolutely literally, one obtains a solution by defining $u(\mathbf{x}, t)$ to be $f(\mathbf{x})$ when $t = 0$ and 0 when $t > 0$, but clearly this is not what is really wanted unless f is identically zero! Rather, in such boundary value problems there is always an implicit continuity assumption: we ask not only that $u(\mathbf{x}, 0) = f(\mathbf{x})$ but that $u(\mathbf{x}, t)$ should approach $f(\mathbf{x})$ as $t \to 0$. The precise way in which this approach is achieved (pointwise convergence, uniform convergence, mean square convergence, etc.) will depend on the particular problem at hand. This is not a matter that requires a lot of deep thought — merely a little care to avoid making silly mistakes.

The wave, heat, and Laplace equations can be generalized by adding in an extra term, as follows:

$$u_{tt} - c^2\nabla^2 u = F(\mathbf{x}, t), \tag{1.10}$$
$$u_t - k\nabla^2 u = F(\mathbf{x}, t), \tag{1.11}$$
$$\nabla^2 u = F(\mathbf{x}). \tag{1.12}$$

These equations are called the **inhomogeneous wave, heat,** and **Laplace equations**; equation (1.12) is also called the **Poisson equation**. Here F is a function that is given in advance, and the original equations (1.2), (1.3), and (1.4) are the special cases where $F = 0$. The interpretation of F will vary with the particular situation considered. In the wave equation (1.10), F may represent a force that is driving the waves; in the case of electromagnetic fields, it represents the effect of charges or currents (see Appendix 1). In the heat equation (1.11), F may represent a source (or sink) of heat within the material in which the heat is flowing. The Poisson equation (1.12) is satisfied by electrostatic potential in a region when F is interpreted as -4π times the charge density in the region, or by the graviatational potential when F is interpreted as 4π times the mass density. (See Appendix 1. The difference in signs occurs because positive masses attract each other, whereas positive charges repel.) The boundary conditions appropriate for these inhomogeneous equations are much the same as for the corresponding homogeneous equations.

Finally, we mention one other basic equation of physics, the **Schrödinger equation**:

$$i\hbar u_t = -\frac{\hbar^2}{2m}\nabla^2 u + V(\mathbf{x})u.$$

In this equation u is the quantum-mechanical wave function for a particle of mass m moving in a potential $V(\mathbf{x})$, $\hbar$ is Planck's constant, and $i = \sqrt{-1}$. When the particle has a definite energy E, the time dependence drops out and one obtains the steady-state equation

$$-\frac{\hbar^2}{2m}\nabla^2 u + V(\mathbf{x})u = Eu.$$

For the physics behind these equations we refer the reader to books on quantum mechanics such as Messiah [39] and Landau-Lifshitz [35]. Readers who are not familiar with this subject can safely ignore the occasional references to the Schrödinger equation, but those who are will find the solutions to some important special cases in later chapters.

EXERCISES

1. Show that $u(x,t) = t^{-1/2}\exp(-x^2/4kt)$ satisfies the heat equation $u_t = ku_{xx}$ for $t > 0$.
2. Show that $u(x,y,t) = t^{-1}\exp[-(x^2+y^2)/4kt]$ satisfies the heat equation $u_t = k(u_{xx}+u_{yy})$ for $t > 0$.
3. Show that $u(x,y) = \log(x^2+y^2)$ satisfies Laplace's equation $u_{xx}+u_{yy} = 0$ for $(x,y) \neq (0,0)$.
4. Show that $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ satisfies Laplace's equation $u_{xx} + u_{yy}+u_{zz} = 0$ for $(x,y,z) \neq (0,0,0)$.
5. Proportionality constants in the equations of physics can often be eliminated by a suitable choice of units of measurement. Mathematically, this amounts to rewriting the equation in terms of new variables that are constant multiples of the original ones. Show that the substitutions $\tau = kt$ and $\tau = ct$ reduce the heat and wave equations, respectively, to $u_\tau = \nabla^2 u$ and $u_{\tau\tau} = \nabla^2 u$.

6. The object of this exercise is to derive *d'Alembert's formula* for the general solution of the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$.
   a. Show that if $u(y,z) = f(y) + g(z)$ where f and g are $C^{(2)}$ functions of one variable, then u satisfies $u_{yz} = 0$. Conversely, show that every $C^{(2)}$ solution of $u_{yz} = 0$ is of this form. (Hint: If $v_y = 0$, then v is independent of y.)
   b. Let $y = x - ct$ and $z = x + ct$. Use the chain rule to show that $u_{tt} - c^2 u_{xx} = -4c^2 u_{yz}$.
   c. Conclude that the general $C^{(2)}$ solution of the wave equation $u_{tt} = c^2 u_{xx}$ is $u(x,t) = f(x - ct) + g(x + ct)$ where f and g are $C^{(2)}$ functions of one variable. (Observe that $f(x - ct)$ represents a wave traveling to the right with speed c, and $g(x + ct)$ represents a wave traveling to the left with speed c.)
   d. Show that the solution of the initial value problem

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x)$$

   is

$$u(x,t) = \tfrac{1}{2}\left[\phi(x - ct) + \phi(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy.$$

7. The voltage v and current i in an electrical cable along the x-axis satisfy the coupled equations

$$i_x + Cv_t + Gv = 0, \qquad v_x + Li_t + Ri = 0,$$

   where C, G, L, and R are the capacitance, (leakage) conductance, inductance, and resistance per unit length in the cable. Show that v and i both satisfy the **telegraph equation**

$$u_{xx} = LCu_{tt} + (RC + LG)u_t + RGu.$$

8. Set $u(x,t) = f(x,t)e^{at}$ in the telegraph equation of Exercise 7. What is the differential equation satisfied by f? Show that a can be chosen so that this equation is of the form $f_{xx} = Af_{tt} + Bf$ (with no first-order term), provided that $LC \neq 0$.

1.2 Linear differential operators

The partial differential equations considered in the preceding section can all be written in the form $L(u) = F$, where $L(u)$ stands for $u_{tt} - c^2\nabla^2 u$, $u_t - k\nabla^2 u$, or $\nabla^2 u$. In each case $L(u)$ is a function obtained from u by performing certain operations involving partial derivatives, which we regard as the result of applying the *operator* L to the function u.

In general, a **linear partial differential operator** L is an operation that transforms a function u of the variables $\mathbf{x} = (x_1, \ldots, x_n)$ into another function $L(u)$ given by

$$L(u) = a(\mathbf{x})u + \sum_{i=1}^{n} b_i(\mathbf{x})\frac{\partial u}{\partial x_i} + \sum_{i,j=1}^{n} c_{ij}(\mathbf{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + \cdots.$$

(Here the dots at the end indicate higher-order terms, but it is understood that the whole sum contains only *finitely* many terms.) In other words, $L(u)$ is obtained by taking a finite collection of partial derivatives of u, multiplying them by the *coefficients* a, b_i, c_{ij}, etc., and adding them up. We may describe the operator L by itself, without reference to an input function u, by writing

$$L = a(\mathbf{x}) + \sum_{i=1}^{n} b_i(\mathbf{x})\frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} c_{ij}(\mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j} + \cdots. \tag{1.13}$$

The term *linear* in the phrase "linear partial differential operator" refers to the following fundamental property: if L is given by (1.13), $u_1, \ldots, u_k$ are any functions possessing the requisite derivatives, and $c_1, \ldots, c_k$ are any constants, then

$$L(c_1 u_1 + \cdots + c_k u_k) = c_1 L(u_1) + \cdots + c_k L(u_k). \tag{1.14}$$

This is an immediate consequence of the fact that the derivative of a sum is the sum of the derivatives, and the derivative of a constant multiple of a function is the constant multiple of the derivative. Any function of the form $c_1 u_1 + \cdots + c_k u_k$ (where the c_j's are constants) is called a **linear combination** of $u_1, \ldots, u_k$. Thus, (1.14) says that L takes every linear combination of u_j's into the corresponding linear combination of $L(u_j)$'s.

More generally, any operator L, differential or otherwise, that satisfies (1.14) is called **linear**; here the inputs u and the outputs $L(u)$ can be any sort of objects for which linear combinations make sense, such as functions, vectors, numbers, etc. For instance, the formula $L(f) = \int_a^b f(t)\,dt$ defines a linear operation taking continuous functions on the interval $[a, b]$ to numbers; and if $\mathbf{x}_0$ is a fixed 3-dimensional vector, the formula $L(\mathbf{x}) = \mathbf{x} \times \mathbf{x}_0$ (the cross product of $\mathbf{x}$ with $\mathbf{x}_0$) defines a linear operation on 3-dimensional vectors.

A **linear partial differential equation** is simply an equation of the form

$$L(u) = F,$$

where L is a linear partial differential operator and F is a function of $\mathbf{x}$. Such an equation is called **homogeneous** if $F = 0$ and **inhomogeneous** if $F \neq 0$. The boundary conditions we associate to a differential equation are usually of a similar form themselves; that is, they are of the form "$B(u) = f$ on the boundary" where B is another linear differential operator and f is a function on the boundary. (We shall often omit the phrase "on the boundary" and write the boundary conditions simply as $B(u) = f$. Here also, the terms *homogeneous* and *inhomogeneous* refer to the cases $f = 0$ and $f \neq 0$.) The linearity of the operators L and B can be restated in the following way.

The Superposition Principle. *If $u_1, \ldots, u_k$ satisfy the linear differential equations $L(u_j) = F_j$ and the boundary conditions $B(u_j) = f_j$ for $j = 1, \ldots, k$, and $c_1, \ldots, c_k$ are any constants, then $u = c_1 u_1 + \cdots + c_k u_k$ satisfies*

$$L(u) = c_1 F_1 + \cdots + c_k F_k, \qquad B(u) = c_1 f_1 + \cdots + c_k f_k.$$

The importance of the superposition principle can hardly be overestimated. We shall use it repeatedly in a number of different ways, of which the most important are the following.

Suppose we want to find all solutions of a differential equation subject to one or more boundary conditions, say

$$L(u) = F, \qquad B(u) = f. \tag{1.15}$$

If we can find all solutions of the corresponding *homogenous* problem

$$L(u) = 0, \qquad B(u) = 0 \tag{1.16}$$

which is often simpler to handle, then it suffices to obtain just *one* solution, say v, of the original problem (1.15). Indeed, if u is any other solution of (1.15), then $w = u - v$ satisfies (1.16), for $L(w) = F - F = 0$ and $B(w) = f - f = 0$. Hence we obtain the general solution of (1.15) by adding the general solution w of (1.16) to any particular solution of (1.15).

In the same spirit, the superposition principle can be used to break down a problem involving several inhomogeneous terms into (presumably simpler) problems in which these terms are dealt with one at a time. For instance, suppose we want to find a solution to (1.15). It suffices to find solutions u_1 and u_2 to the problems

$$L(u_1) = F, \qquad B(u_1) = 0;$$
$$L(u_2) = 0, \qquad B(u_2) = f,$$

for we can then take $u = u_1 + u_2$.

Perhaps most important, if $u_1, u_2, \ldots$ are any solutions to a *homogeneous* differential equation $L(u) = 0$ that satisfy *homogeneous* boundary conditions $B(u) = 0$, then any linear combination of the u_j's will satisfy the same differential equation and the same boundary conditions. Thus, starting out with a sequence of solutions u_j, we can generate many more solutions by taking linear combinations. If we then take appropriate *limits* of such linear combinations, we arrive at solutions defined by infinite series or integrals — and this is where things get interesting!

Of course, there are also *nonlinear* differential equations involving *nonlinear* operations such as $L(u) = u_{xx} - \sin u$ or $L(u) = uu_x + (u_y)^3$. Indeed, many of the important equations of physics and engineering, including most of the refinements of the wave equation to describe waves and vibrations, are nonlinear.

However, nonlinear equations are, on the whole, much more difficult to solve than linear ones, and their study is beyond the scope of this book.

One final note: the reader will have observed that all the differential equations we discussed in §1.1 involve the Laplacian ∇^2. The reason for this is that the Laplacian commutes with all rigid motions of Euclidean space; that is, if $\mathcal{T}$ denotes any translation or rotation of n-space, then $\nabla^2(f \circ \mathcal{T}) = (\nabla^2 f) \circ \mathcal{T}$ for all functions f. Moreover, the *only* linear differential operators of order ≤ 2 that have this property are the operators $a\nabla^2 + b$ where a and b are constants. Hence, the differential equation describing any process that is spatially symmetric (i.e., unaffected by translations and rotations) is likely to involve the Laplacian.

EXERCISES

1. Suppose u_1 and u_2 are both solutions of the linear differential equation $L(u) = f$, where $f \neq 0$. Under what conditions is the linear combination $c_1u_1 + c_2u_2$ also a solution of this equation?
2. Consider the nonlinear (ordinary) differential equation $u' = u(1-u)$.
   a. Show that $u_1(x) = e^x/(1+e^x)$ and $u_2(x) = 1$ are solutions.
   b. Show that $u_1 + u_2$ is not a solution.
   c. For which values of c is cu_1 a solution? How about cu_2?
3. Give examples of linear differential operators L and M for which it is not true that $L(M(u)) = M(L(u))$ for all u. (Hint: At least one of L and M must have nonconstant coefficients.)
4. What form must G have for the differential equation $u_{tt} - u_{xx} = G(x,t,u)$ to be linear? Linear and homogeneous?
5. a. Show that for $n = 1, 2, 3, \ldots$, $u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$ satisfies

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(1,y) = u(x,0) = 0.$$

   b. Find a linear combination of the u_n's that satisfies $u(x,1) = \sin 2\pi x - \sin 3\pi x$.
   c. Show that for $n = 1, 2, 3, \ldots$, $\widetilde{u}_n(x,y) = \sin(n\pi x)\sinh n\pi(1-y)$ satisfies

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(1,y) = u(x,1) = 0.$$

   d. Find a linear combination of the $\widetilde{u}_n$'s that satisfies $u(x,0) = 2\sin \pi x$.
   e. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = 0,$$
$$u(x,0) = 2\sin \pi x, \quad u(x,1) = \sin 2\pi x - \sin 3\pi x.$$

1.3 Separation of variables

In this section we discuss a very useful technique for solving certain linear partial differential equations, known as *separation of variables*. This technique works only for very special sorts of equations, but fortunately the equations for which it works include many of the most important ones.

The idea is as follows. Suppose, for simplicity, that we have a homogeneous partial differential equation $L(u) = 0$ involving just two independent variables x and y, with some homogeneous boundary conditions $B(u) = 0$. We try to find solutions u of the form

$$u(x, y) = X(x)Y(y).$$

If the method is to work, when we substitute this formula for u into the equation $L(u) = 0$, the terms can be rearranged so that the left side of the equation involves only the variable x and the right side involves only the variable y, say $P(x) = Q(y)$. But since x and y are independent, a quantity that depends on x alone and also on y alone must be a constant. Hence we have $P(x) = C$ and $Q(y) = C$, and these equations will be *ordinary* differential equations for the functions X and Y whose product is u. With luck, these equations can be solved subject to the boundary conditions on X and Y that are implied by the original conditions on u, and we thus obtain a whole family of solutions by varying the constant C. By the superposition principle, all linear combinations of these will also be solutions; and if we are lucky, we will obtain *all* solutions of the original problem by taking appropriate limits of these linear combinations.

The same procedure can be used for equations for functions of more than two variables. If there are three independent variables involved, say x, y, and z, we look for solutions of the form u of the form

$$u(x, y, z) = X(x)v(y, z).$$

If the variables can be separated, we obtain an *ordinary* differential equation for X and a *partial* differential equation for v, but now involving only the *two* variables y and z. We can then try to write $v(y, z) = Y(y)Z(z)$ and obtain ordinary differential equations for Y and Z. In other words, we use separation of variables to "peel off" the independent variables one at a time, thereby reducing the original problem to some simpler ones.

Of course, once one has reduced the problem to some ordinary differential equations, one must be able to solve them! For the time being all our examples will involve homogeneous equations with real constant coefficients, whose solutions we now briefly review. (See, for example, Boyce-DiPrima [10] for a more extensive discussion.) For first-order equations the situation is very simple:

$$f' = af \quad \Longrightarrow \quad f(x) = Ce^{ax}.$$

For second-order equations, the basic fact is as follows.

Theorem 1.1. *The general solution of* $f'' + af' + bf = 0$ *is* $f(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$, *where* r_1, r_2 *are the roots of the equation* $r^2 + ar + b = 0$ *and* C_1, C_2 *are arbitrary complex numbers. If* $r_1 = r_2$, *the general solution is* $(C_1 + C_2 x)e^{r_1 x}$.

Here r_1 and r_2 may, of course, be complex; see Appendix 2 for a discussion of the complex exponential function. In certain cases it may be more convenient to express the solution in terms of trigonometric or hyperbolic functions. In particular:

(i) *If* $r_1 = \rho + i\sigma$ *and* $r_2 = \rho - i\sigma$, *the general solution is* $e^{\rho x}(C_1 \cos \sigma x + C_2 \sin \sigma x)$.

(ii) *If* $\alpha > 0$, *the general solution of* $f'' + \alpha^2 f = 0$ *is* $C_1 \cos \alpha x + C_2 \sin \alpha x$, *and the general solution of* $f'' - \alpha^2 f = 0$ *is* $C_1 \cosh \alpha x + C_2 \sinh \alpha x$.

Enough generalities; let us look at a couple of specific examples.

Consider the problem of 1-dimensional heat flow: we may think of a circular metal rod of length l, insulated along its curved surface so that heat can enter or leave only at the ends. Suppose, moreover, that both ends are held at temperature zero. (Zero in which temperature scale? It doesn't matter: the mathematics is the same.) Ignoring the question of initial conditions for the moment, we then have the boundary value problem

$$u_t = ku_{xx}, \qquad u(0,t) = u(l,t) = 0. \tag{1.17}$$

If we substitute $u(x,t) = X(x)T(t)$ into (1.17), we obtain

$$X(x)T'(t) = kX''(x)T(t), \tag{1.18}$$

$$X(0) = X(l) = 0. \tag{1.19}$$

The variables in (1.18) may be separated by dividing both sides by $kX(x)T(t)$, yielding

$$T'(t)/kT(t) = X''(x)/X(x).$$

Now the left side depends only on t, whereas the right side depends only on x; since they are equal, they must both be equal to a constant A:

$$T'(t) = AkT(t), \qquad X''(x) = AX(x).$$

These are simple ordinary differential equations for T and X that can be solved by elementary methods — indeed, almost by inspection. The general solution of the equation for T is

$$T(t) = C_0 e^{Akt},$$

and the general solution of the equation for X is

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x, \qquad \lambda = \sqrt{-A}. \tag{1.20}$$

(If A is positive, one might prefer to avoid imaginary numbers by rewriting (1.20) as

$$X(x) = C_1' \cosh \mu x + C_2' \sinh \mu x = \frac{C_1' + C_2'}{2} e^{\mu x} + \frac{C_1' - C_2'}{2} e^{-\mu x}, \qquad \mu = \sqrt{A}.$$

But so far A is just an arbitrary (possibly complex) constant, so there is no reason yet to choose one form over the other.) However, we must now take account of the boundary conditions (1.19). The condition $X(0) = 0$ forces $C_1 = 0$ in (1.20), and the condition $X(l) = 0$ then becomes $C_2 \sin \lambda l = 0$. If we take $C_2 = 0$, then our solution $u(x, t)$ vanishes identically, which is of no interest: we are looking for *nontrivial* solutions. So we take $C_2 \neq 0$; hence $\sin \lambda l = 0$, which means that $\lambda l = n\pi$ for some integer n; in other words, $A = -(n\pi/l)^2$. (So A is negative after all!) We may take $n > 0$, since the case $n = 0$ gives the zero solution and replacing n by $-n$ merely amounts to replacing C_2 by $-C_2$.

In short, for every positive integer n we have obtained a solution $u_n(x, t)$ of (1.17), namely,

$$u_n(x,t) = \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \sin \frac{n\pi x}{l} \qquad (n = 1, 2, 3, \ldots).$$

(We have taken $C_0 = C_2 = 1$; other choices of C_0 and C_2 give constant multiples of u_n.) We obtain more solutions by taking linear combinations of the u_n's, and then passing to *infinite* linear combinations — that is, infinite series

$$u = \sum_1^\infty a_n u_n = \sum_1^\infty a_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \sin \frac{n\pi x}{l}. \tag{1.21}$$

Of course, there are questions to be answered about the convergence of such series, but for the moment we shall not worry about that.

Finally, we bring the initial conditions into the picture: can we solve (1.17) subject to the initial condition $u(x, 0) = f(x)$, where f is a given function on the interval $(0, l)$? The solution (1.21) will do the job, provided that

$$f(x) = \sum_1^\infty a_n \sin \frac{n\pi x}{l}. \tag{1.22}$$

We have now arived at one of the main subjects of this book: the study of series expansions like (1.22). Before setting foot in this new territory, however, let us look at a couple of other boundary value problems.

Consider the problem of heat flow in a rod, as before, but now assume that the ends of the rod are insulated. Thus, instead of (1.17) we consider

$$u_t = k u_{xx}, \qquad u_x(0, t) = u_x(l, t) = 0. \tag{1.23}$$

The technique we used to solve (1.17) also works here, with only the following differences. The conditions (1.19) are replaced by

$$X'(0) = X'(l) = 0, \tag{1.24}$$

which force $C_2 = 0$ (rather than $C_1 = 0$) and $\lambda l = n\pi$ in (1.20). Again, we may assume that $n \geq 0$ since $\cos(n\pi x/l) = \cos(-n\pi x/l)$, but now we must include $n = 0$. We thus obtain the sequence of solutions

$$u_n(x,t) = \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right)\cos\frac{n\pi x}{l} \qquad (n = 0, 1, 2, \ldots),$$

which can be combined to form the series

$$u = \sum_0^\infty a_n u_n = \sum_0^\infty a_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right)\cos\frac{n\pi x}{l}.$$

This series will solve the problem (1.23) subject to the initial condition $u(x,0) = f(x)$ provided that

$$f(x) = \sum_0^\infty a_n \cos\frac{n\pi x}{l}. \tag{1.25}$$

Thus we have arrived at another series expansion problem, different from but similar to (1.22).

For yet another variation on the same theme, consider heat flow in a rod that is bent into the shape of a circle, with the ends joined together. We may specify the position of a point on the circle by its angular coordinate θ, measured from some fixed base point. Since linear distance on a circle is proportional to angular distance ($\Delta x = r\Delta\theta$ where r is the radius), the heat equation $u_t = k_0 u_{xx}$ can be rewritten as

$$u_t = ku_{\theta\theta}$$

where $k = k_0/r^2$. We try to find solutions of the form $u(\theta,t) = \Theta(\theta)T(t)$, and just as before we find that

$$T(t) = C_0 e^{Akt}, \qquad \Theta(\theta) = C_1\cos\theta\sqrt{-A} + C_2\sin\theta\sqrt{-A} \tag{1.26}$$

for some constant A. Here there are no boundary conditions like (1.19) or (1.24) because the rod has no ends. Instead, since the angular coordinate θ is well-defined only up to multiples of 2π, we have the requirement that $\Theta(\theta)$ must be periodic with period 2π. This condition does not kill off either of the coefficients C_1 or C_2 in (1.26), but it does force $\sqrt{-A}$ to be an integer n. The upshot is that we obtain series solutions of the form

$$u(\theta,t) = \sum_0^\infty (a_n\cos n\theta + b_n\sin n\theta)e^{-n^2kt},$$

and such a series will satisfy the initial condition $u(\theta,0) = f(\theta)$ provided that

$$f(\theta) = \sum_0^\infty (a_n\cos n\theta + b_n\sin n\theta). \tag{1.27}$$

Finally, we present an illustration of these techniques involving something other than the heat equation. Consider the problem of a vibrating string of length l, fixed at both endpoints. The mathematical problem to be solved is

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(l,t) = 0. \tag{1.28}$$

If we take $u(x,t) = X(x)T(t)$, (1.28) becomes

$$X(x)T''(t) = c^2 X''(x)T(t), \tag{1.29}$$

$$X(0) = X(l) = 0 \tag{1.30}$$

On dividing (1.29) through by $c^2 X(x)T(t)$, we get

$$X''(x)/X(x) = T''(t)/c^2 T(t),$$

and both sides of this equation must be equal to a constant that we call $-\lambda^2$. (As before, λ might be any complex number until we pin it down further.) Hence,

$$X''(x) = -\lambda^2 X(x), \qquad T''(t) = -\lambda^2 c^2 T(t).$$

The general solutions of these ordinary differential equations are

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x, \qquad T(t) = C_3 \cos \lambda ct + C_4 \sin \lambda ct.$$

As with the heat equation, the boundary conditions (1.30) imply that $C_1 = 0$ and $\lambda = n\pi/l$ where n is a (positive) integer. We therefore obtain the series solutions

$$u(x,t) = \sum_1^\infty \sin \frac{n\pi x}{l} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right). \tag{1.31}$$

We recall from §1.1 that the appropriate initial conditions for this problem are to specify $u(x,0) = f(x)$ and $(\partial u/\partial t)(x,0) = g(x)$. Setting $t = 0$ in (1.31), we find that

$$f(x) = \sum_1^\infty a_n \sin \frac{n\pi x}{l},$$

whereas if we differentiate (1.31) with respect to t (ignoring possible difficulties about differentiating an infinite series term by term) and then set $t = 0$, we get

$$g(x) = \sum_1^\infty \frac{n\pi c}{l} b_n \sin \frac{n\pi x}{l}.$$

Thus we are led once again to the problem of expanding f and g in a sine series of the form (1.22).

To sum up: in order to carry out the program of solving differential equations by separation of variables, there are two problems that have to be addressed. First, there are some technicalities connected with the convergence properties of infinite series; these are sometimes annoying but rarely are really serious. Second and more important, the following questions must be answered. *Can a given function on the interval* $(0,l)$ *be expanded in a sine series* (1.22) *or a cosine series* (1.25)? *Can a periodic function with period* 2π *be expanded in a series of the form* (1.27)? *If so, how?*

It is to these and related questions that the next chapter is devoted.

EXERCISES

1. Derive pairs of ordinary differential equations from the following partial differential equations by separation of variables, or show that it is not possible.
   a. $yu_{xx} + u_y = 0$.
   b. $x^2u_{xx} + xu_x + u_{yy} + u = 0$.
   c. $u_{xx} + u_{xy} + u_{yy} = 0$.
   d. $u_{xx} + u_{xy} + u_y = 0$.
2. Derive sets of three ordinary differential equations from the following partial differential equations by separation of variables.
   a. $yu_{xx} + xu_{yy} + xyu_{zz} = 0$.
   b. $x^2u_{xx} + xu_x + u_{yy} + x^2u_{zz} = 0$.
3. Use the results in the text to solve

$$u_{tt} = 9u_{xx}, \qquad u(0,t) = u(1,t) = 0,$$
$$u(x,0) = 2\sin\pi x - 3\sin 4\pi x, \qquad u_t(x,0) = 0 \qquad (0 < x < 1).$$

4. Use the results in the text to solve

$$u_t = \tfrac{1}{10}u_{xx}, \qquad u_x(0,t) = u_x(\pi,t) = 0,$$
$$u(x,0) = 3 - 4\cos 2x \qquad (0 < x < \pi).$$

   Determine a value of t_0 so that $|u(x,t) - 3| < 10^{-4}$ for $t > t_0$.
5. By separation of variables, derive the solutions $u_n(x,y) = \sin n\pi x \sinh n\pi y$ of

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(1,y) = u(x,0) = 0$$

   that were discussed in Exercise 5a, §1.2.
6. By separation of variables, derive the family

$$u_{mn}^{\pm}(x,y,z) = \sin m\pi x \cos n\pi y \exp\left(\pm\sqrt{m^2+n^2}\,\pi z\right)$$

   of the problem

$$\nabla^2 u = 0, \qquad u(0,y,z) = u(1,y,z) = u_y(x,0,z) = u_y(x,1,z) = 0.$$

7. Use separation of variables to find an infinite family of independent solutions of

$$u_t = ku_{xx}, \qquad u(0,t) = 0, \qquad u_x(l,t) = 0,$$

   representing heat flow in a rod with one end held at temperature zero and the other end insulated.

CHAPTER 2
FOURIER SERIES

In Chapter 1 we derived three problems concerning the expansion of functions in terms of sines and cosines. The most fundamental of these is the expansion of periodic functions, which is of importance not only for boundary value problems but for the analysis of any sort of periodic phenomena, and which has provided either direct or indirect inspiration for many of the developments of modern mathematical analysis. Most of this chapter is devoted to the study of periodic functions. Once they are understood, the other two expansion problems of §1.3 can be solved without difficulty, as we shall see in §2.4.

In many respects it is simpler and neater to work with the complex exponential function $e^{i\theta}$ instead of the trigonometric functions $\cos\theta$ and $\sin\theta$. We recall that these functions are related by the formulas

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i},$$
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The advantages of cosine and sine are that they are real-valued and are, respectively, even and odd; the advantages of the exponential are that its differentiation formula $(e^{i\theta})' = ie^{i\theta}$ and addition formula $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$ are simpler than the corresponding formulas for cosine and sine. Accordingly, it is worthwhile to be able to translate one formulation into the other without much effort; we urge the readers who have not yet acquired this facility to spend a little time doing so. A more complete list of the properties of exponential and trigonometric functions of complex variables will be found in Appendix 2.

2.1 The Fourier series of a periodic function

Suppose that $f(\theta)$ is a function defined on the real line such that $f(\theta+2\pi) = f(\theta)$ for all θ. Such functions are said to be **periodic with period** 2π, or 2π-**periodic** for short. We shall assume that f is Riemann integrable on every bounded interval; this will be the case if f is bounded and is continuous except perhaps at finitely many points in each bounded interval. (We shall consider various other hypotheses on f in subsequent sections.) Since we shall be using the complex exponential function, we shall allow f to be complex-valued rather than merely real-valued. This bit of extra generality causes no additional difficulties and indeed simplifies some things; moreover, in more advanced work it is often crucial to use complex functions.

We wish to know if f can be expanded in a series

$$f(\theta) = \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta). \tag{2.1}$$

Here $\frac{1}{2}a_0$ is the coefficient of the constant function $1 = \cos 0\theta$, and the factor of $\frac{1}{2}$ is incorporated in it for reasons of later convenience (see the remark following equation (2.6)). There is no b_0 because $\sin 0\theta = 0$.

In view of the formulas $\cos n\theta = (e^{in\theta} + e^{-in\theta})/2$ and $\sin n\theta = (e^{in\theta} - e^{-in\theta})/2i$, (2.1) can be rewritten as

$$f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} \tag{2.2}$$

where

$$c_0 = \tfrac{1}{2}a_0; \qquad c_n = \tfrac{1}{2}(a_n - ib_n) \text{ and } c_{-n} = \tfrac{1}{2}(a_n + ib_n) \text{ for } n = 1, 2, 3, \dots \tag{2.3}$$

Alternatively, if we start out with (2.2), by using the formulas $e^{in\theta} = \cos n\theta + i \sin n\theta$, $\cos(-n)\theta = \cos n\theta$, and $\sin(-n)\theta = -\sin n\theta$, we can put it in the form (2.1) where

$$a_0 = 2c_0; \qquad a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}) \quad \text{for } n = 1, 2, 3, \dots \tag{2.4}$$

In what follows we shall work primarily with (2.2), but we shall also show how to interpret the results in terms of (2.1).

As a first step towards analyzing general periodic functions in terms of trigonometric series, let us consider the following question. If we know to begin with that $f(\theta)$ has a series expansion of the form (2.2), how can the coefficients c_n be calculated in terms of f? The answer to this question is appealingly simple. Let us multiply both sides of (2.2) by $e^{-ik\theta}$ (k being an integer) and integrate from $-\pi$ to π. Taking on faith for the moment that it is permissible to integrate the series term by term, we obtain

$$\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\,d\theta = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)\theta}\,d\theta.$$

But

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}\,d\theta = \frac{1}{i(n-k)} e^{i(n-k)\theta}\Big|_{-\pi}^{\pi} = \frac{(-1)^{n-k} - (-1)^{n-k}}{i(n-k)} = 0 \qquad \text{if } n \neq k,$$

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}\,d\theta = \int_{-\pi}^{\pi} d\theta = 2\pi \qquad \text{if } n = k.$$

Hence the only term in the series that survives the integration is the term with $n = k$, and we obtain

$$\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\, d\theta = 2\pi c_k.$$

In other words, relabeling the integer k as n, we have the desired formula for the coefficients c_n:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta. \tag{2.5}$$

It is now an easy matter to find the coefficients a_n and b_n for the series (2.1):

$$a_0 = 2c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta,$$

and for $n = 1, 2, 3, \ldots,$

$$a_n = c_n + c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(e^{-in\theta} + e^{in\theta})\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta,$$

$$b_n = i(c_n - c_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(\theta)(e^{-in\theta} - e^{in\theta})\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta;$$

that is,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta \qquad (n \geq 0); \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta \qquad (n \geq 1). \end{aligned} \tag{2.6}$$

(Note that the formula for a_n here holds also for $n = 0$; this is the reason for the factor of $\frac{1}{2}$ in (2.1).)

To recapitulate: *if* f has a series expansion of the form (2.1) (or (2.2)), and *if* the series converges decently so that term-by-term integration is permissible, *then* the coefficients a_n and b_n [or c_n] are given by (2.6) [or (2.5)]. But now if f is *any* Riemann-integrable periodic function, the integrals in (2.5) and (2.6) make perfectly good sense, and we can use them to *define* the coefficients a_n, b_n, and c_n. We are now in a position to make a formal definition.

Definition. Suppose f is periodic with period 2π and integrable over $[-\pi, \pi]$. The numbers c_n defined by (2.5), or the numbers a_n and b_n defined by (2.6), are called the **Fourier coefficients** of f, and the corresponding series

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} \quad \text{or} \quad \tfrac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

is called the **Fourier series** of f.

Instead of integrating from $-\pi$ to π in (2.5) and (2.6), one could equally well integrate over any interval of length 2π, for instance from 0 to 2π. The result will be the same since the integrands are all 2π-periodic. This is an instance of the following general fact, which is sufficiently useful to merit a special mention.

Lemma 2.1. *If F is periodic with period P, then $\int_a^{a+P} F(x)\,dx$ is independent of a.*

Proof: Let

$$g(a) = \int_a^{a+P} F(x)\,dx = \int_0^{a+P} F(x)\,dx - \int_0^a F(x)\,dx.$$

By the fundamental theorem of calculus, $g'(a) = F(a+P) - F(a)$, so by the periodicity of F, g' vanishes identically. Thus g is constant. ∎

Another useful observation in this context is that

$$\int_{-a}^{a} F(x)\,dx = \begin{cases} 2\int_0^a F(x)\,dx & \text{if } F \text{ is even,} \\ 0 & \text{if } F \text{ is odd.} \end{cases}$$

(Recall that F is **even** if $F(-x) = F(x)$ and **odd** if $F(-x) = -F(x)$.) Since $\cos n\theta$ is even and $\sin n\theta$ is odd, we have the following result.

Lemma 2.2. *With reference to the formulas (2.6),*

$$\text{if } f \text{ is even,} \quad a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta\,d\theta \quad \text{and} \quad b_n = 0;$$

$$\text{if } f \text{ is odd,} \quad a_n = 0 \quad \text{and} \quad b_n = \frac{2}{\pi}\int_0^\pi f(\theta)\sin n\theta\,d\theta.$$

Whether the Fourier series of a 2π-periodic function f is written in the trigonometric form (2.1) or the exponential form (2.2), the constant term in the series is

$$c_0 = \tfrac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta,$$

which is nothing but the average or mean value of f on the interval $[-\pi, \pi]$. By Lemma 2.1, it is also the mean value of f on *any* interval of length 2π. This fact is very useful, and it may be more easily remembered than the integral formula; accordingly, we display it as a lemma.

Lemma 2.3. *The constant term in the Fourier series of a 2π-periodic function f is the mean value of f on an interval of length 2π.*

The preceding discussion shows that if we wish to find a trigonometric series that converges to a given periodic function f, the Fourier series of f is the only reasonable candidate; but we do not yet know whether it always does the job. Before tackling this general question, let us compute a couple of examples.

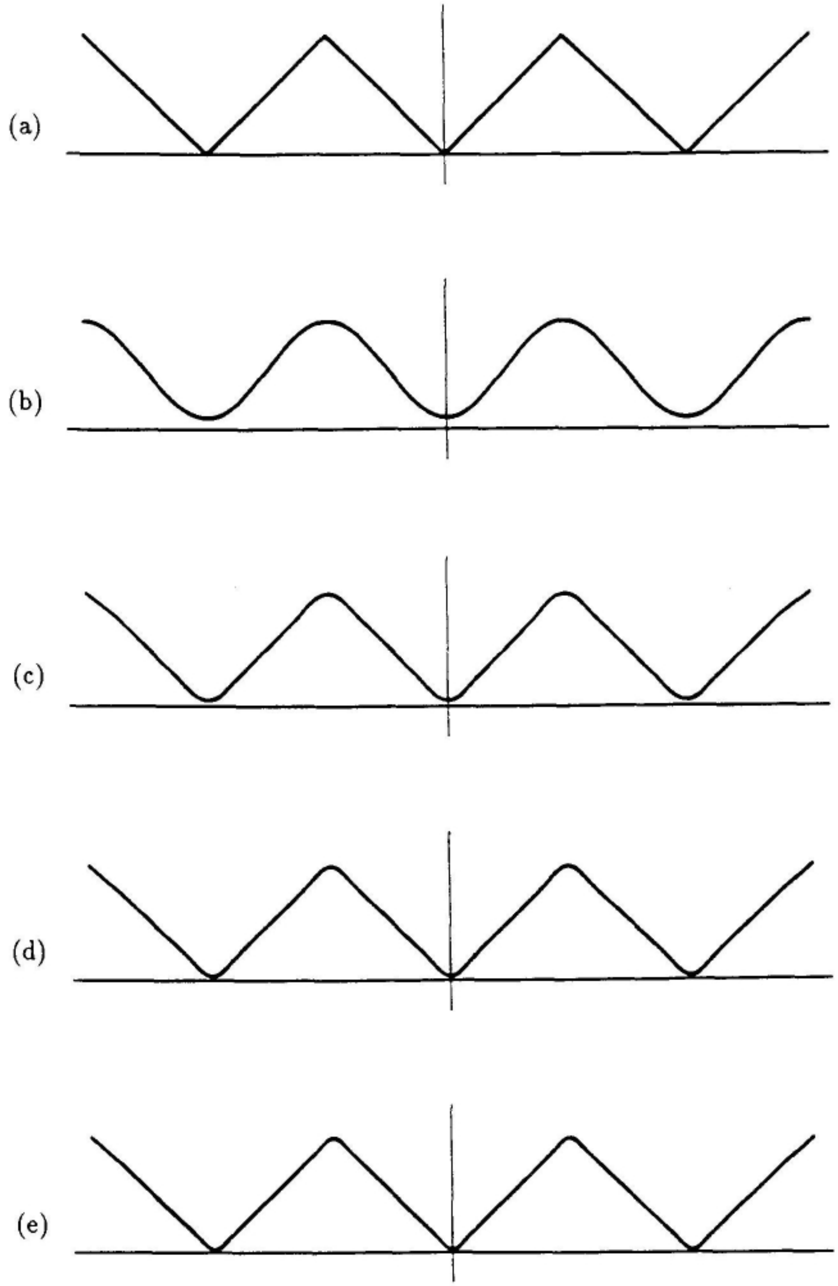


FIGURE 2.1. The triangle wave of Example 1 and some partial sums of its Fourier series: (a) the triangle wave, (b) S_1, (c) S_2, (d) S_3, and (e) S_4, where $S_K = \frac{1}{2}\pi - (4/\pi)\sum_1^K (2k-1)^{-2}\cos(2k-1)\theta$.

Example 1. Let f be the 2π-periodic function determined by the formula

$$f(\theta) = |\theta| \quad \text{for} \ -\pi \leq \theta \leq \pi;$$

that is, f is the triangle wave depicted in Figure 2.1(a). Since f is even, we can calculate the coefficients a_n and b_n by using Lemma 2.2. We have $b_n = 0$ and

$$a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta \, d\theta = \frac{2}{\pi}\int_0^\pi \theta \cos n\theta \, d\theta.$$

Thus, for $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^\pi \theta \, d\theta = \frac{1}{\pi}\theta^2\Big|_0^\pi = \pi,$$

and for $n > 0$,

$$a_n = \frac{2}{\pi}\frac{\theta \sin n\theta}{n}\Big|_0^\pi - \frac{2}{\pi}\int_0^\pi \frac{\sin n\theta}{n} d\theta = \frac{2}{\pi}\frac{\cos n\theta}{n^2}\Big|_0^\pi = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2},$$

since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$. Now, $(-1)^n - 1$ equals -2 when n is odd and 0 when n is even. Therefore, the Fourier series of f is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\ldots} \frac{1}{n^2}\cos n\theta = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\theta}{(2k-1)^2}. \tag{2.7}$$

The graphs of the first few partial sums of this series are shown in Figure 2.1(b–e). Evidently they provide good approximations to f: after only five terms (including the constant term), the graph of the partial sum is almost indistinguishable from the graph of f, except that the corners are a bit rounded. Moreover, we can easily see that the whole series converges absolutely, by comparison to the convergent series $\sum_1^\infty n^{-2}$.

Example 2. Let g be the 2π-periodic function determined by the formula

$$g(\theta) = \theta \quad \text{for} \ -\pi < \theta \leq \pi.$$

In other words, g is the sawtooth wave depicted in Figure 2.2(a). We could use Lemma 2.2 to calculate a_n and b_n since g is odd, but for the sake of variety we shall use (2.5) to calculate c_n instead. For $n = 0$ we have

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta \, d\theta = 0,$$

and for $n \neq 0$ we integrate by parts to obtain

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta} \, d\theta = \frac{1}{2\pi}\frac{\theta e^{-in\theta}}{-in}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-in\theta}}{-in} d\theta$$
$$= \frac{1}{2\pi}e^{-in\theta}\left(\frac{\theta}{-in} + \frac{1}{n^2}\right)\Big|_{-\pi}^{\pi} = \frac{(-1)^{n+1}}{in},$$

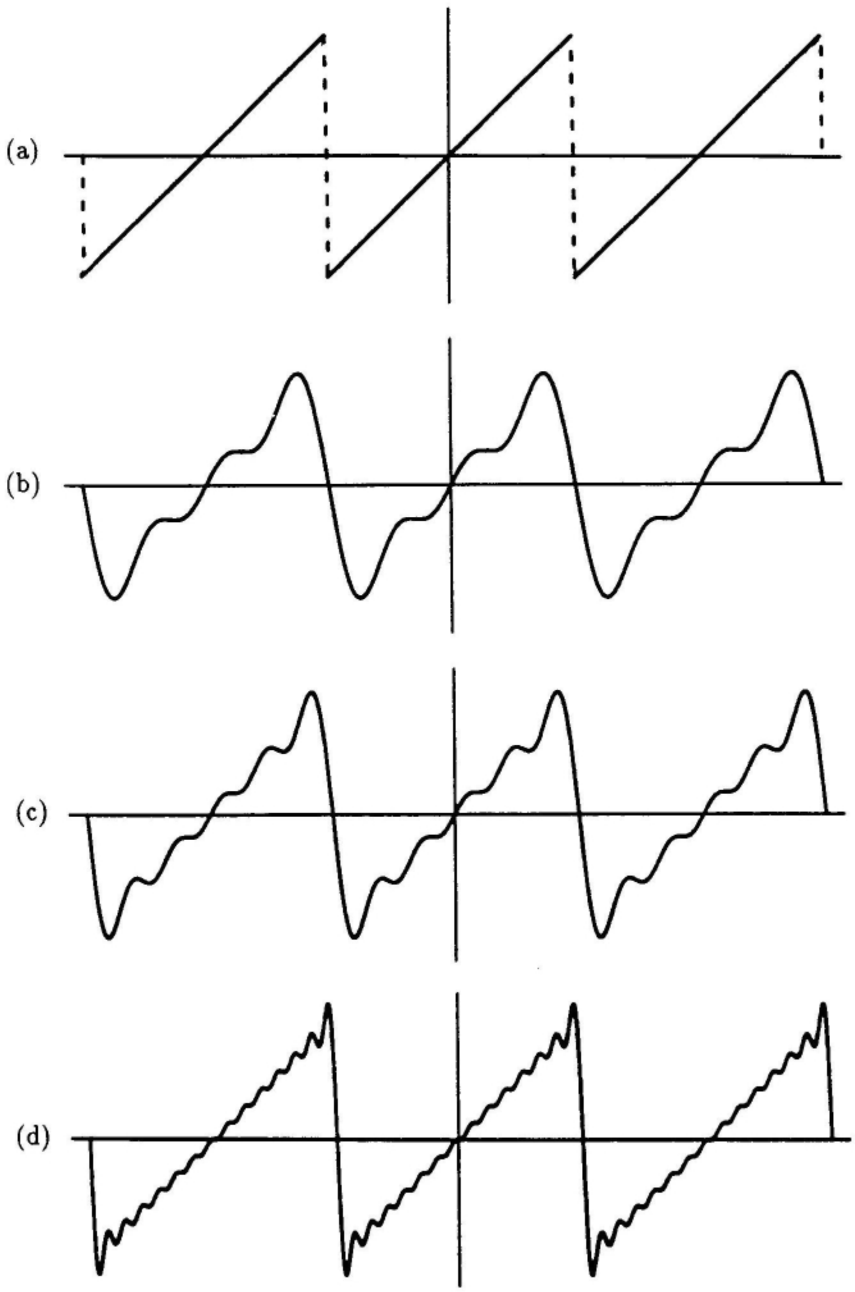


FIGURE 2.2. The sawtooth wave of Example 2 and some partial sums of its Fourier series: (a) the sawtooth wave, (b) S_3, (c) S_5, and (d) S_{14}, where $S_N = 2\sum_1^N (-1)^{n+1} n^{-1} \sin n\theta$.

since $e^{-in\pi} = (-1)^n$. Hence the Fourier series of g is

$$\sum_{n\neq 0} \frac{(-1)^{n+1}}{in} e^{in\theta}.$$

Here n runs through all positive and negative integers. Since $(-1)^n = (-1)^{-n}$, the nth and $(-n)$th terms of this series can be combined to give

$$(-1)^{n+1}\left(\frac{e^{in\theta}}{in} + \frac{e^{-in\theta}}{-in}\right) = \frac{2(-1)^{n+1}}{n}\sin n\theta,$$

and thus the Fourier series of g is

$$2\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}\sin n\theta. \tag{2.8}$$

The graphs of some partial sums of this series are shown in Figure 2.2(b–d). One can see that these partial sums do approximate the original function g, but a comparison of Figures 2.1 and 2.2 shows that the quality of the approximation here is markedly inferior to that in Example 1. One must add many more terms to the series to get a comparably close fit to the original curve, particularly near the discontinuities. (See also Figure 2.8 in §2.6, showing the 40th partial sum of the Fourier series of the reversed sawtooth wave, for an even more dramatic demonstration of this fact.)

Analytically, the reason for this is that the terms in the series (2.7) tend to zero much more rapidly than the terms in the series (2.8). Precisely, if one disregards the even-order terms in (2.7) (which are all zero), the nth term in (2.7) is of the order of magnitude of $(2n-1)^{-2}$, whereas the nth term in (2.8) is of the order of magnitude of n^{-1}. Thus, the contributions of the high-order terms is much less in (2.7) than in (2.8). As we shall see in §2.3, this phenomenon is intimately related to the fact that the triangle wave is smoother than the sawtooth wave: the former is everywhere continuous, whereas the latter has jump discontinuities. The point is that the rougher a function is, the more difficult it is to approximate it with perfectly smooth functions like linear combinations of $\cos n\theta$ and $\sin n\theta$.

In fact, there sems to be some danger that the series (2.8) will not converge: the nth term has magnitude roughly n^{-1} in general, and $\sum_1^\infty n^{-1}$ diverges. On the other hand, at a given point θ some of the functions $\sin n\theta$ will be positive and others will be negative, so there may be some cancellation effects that will prevent divergence. This is in fact the case, as we shall prove in the next section. For the moment, we simply wish to impress on the reader that the convergence of Fourier series is not a simple matter.

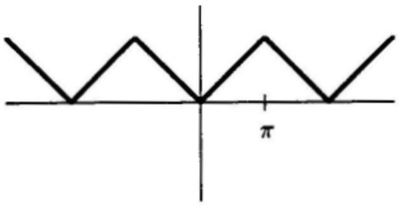
Table 1 gives a list of some elementary Fourier series. It includes all the examples we shall need later on. The fact that all the functions in this table really are the sums of their Fourier series (except perhaps at their points of discontinuity) follows from Theorem 2.1 in §2.2.

We conclude this section by deriving an estimate on the Fourier coefficients that will be needed to establish convergence results in the following sections.

TABLE 1. FOURIER SERIES

The functions f in this table are all understood to be 2π-periodic. The formula for $f(\theta)$ on either $(-\pi, \pi)$ or $(0, 2\pi)$ (except perhaps at its points of discontinuity) is given in the left column; the Fourier series of f is given in the right column; and the graph of f is sketched on the facing page.

1.	$f(\theta) = \theta \quad (-\pi < \theta < \pi)$	$2\sum_1^\infty \frac{(-1)^{n+1}}{n} \sin n\theta$
2.	$f(\theta) = \lvert\theta\rvert \quad (-\pi < \theta < \pi)$	$\frac{\pi}{2} - \frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2}$
3.	$f(\theta) = \pi - \theta \quad (0 < \theta < 2\pi)$	$2\sum_1^\infty \frac{\sin n\theta}{n}$
4.	$f(\theta) = \begin{cases} 0 & (-\pi < \theta < 0) \\ \theta & (0 < \theta < \pi) \end{cases}$	$\frac{\pi}{4} - \frac{2}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2} + \sum_1^\infty \frac{(-1)^{(n+1)}}{n} \sin n\theta$
5.	$f(\theta) = \sin^2\theta$	$\frac{1}{2} - \frac{1}{2}\cos 2\theta$
6.	$f(\theta) = \begin{cases} -1 & (-\pi < \theta < 0) \\ 1 & (0 < \theta < \pi) \end{cases}$	$\frac{4}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{2n-1}$
7.	$f(\theta) = \begin{cases} 0 & (-\pi < \theta < 0) \\ 1 & (0 < \theta < \pi) \end{cases}$	$\frac{1}{2} + \frac{2}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{2n-1}$
8.	$f(\theta) = \lvert\sin\theta\rvert$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2-1}$
9.	$f(\theta) = \lvert\cos\theta\rvert$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{(-1)^n \cos 2n\theta}{4n^2-1}$
10.	$f(\theta) = \begin{cases} 0 & (-\pi < \theta < 0) \\ \sin\theta & (0 < \theta < \pi) \end{cases}$	$\frac{1}{\pi} - \frac{2}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2-1} + \frac{1}{2}\sin\theta$



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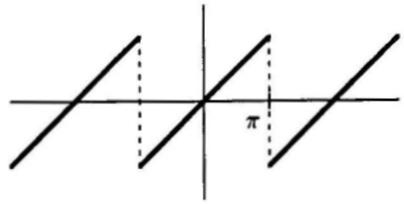

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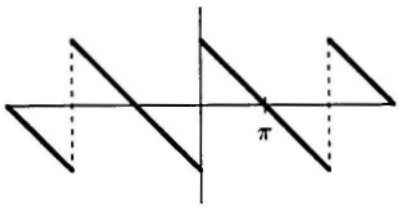

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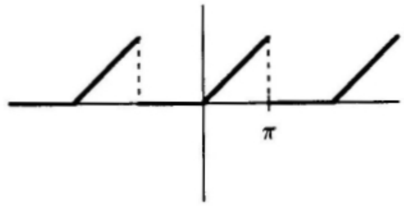

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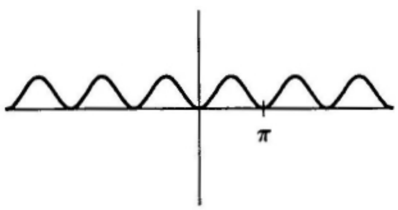

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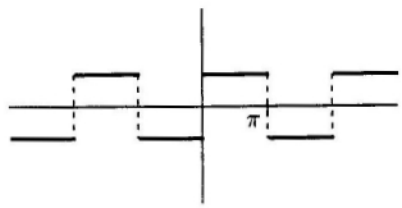

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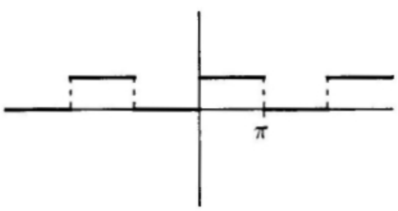

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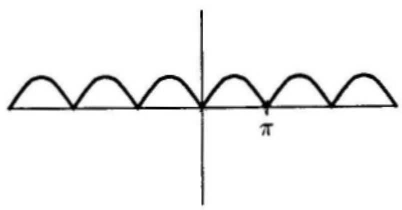

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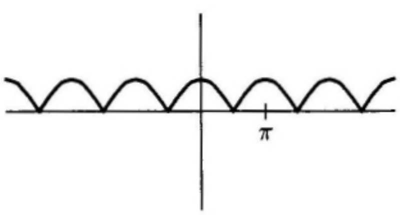

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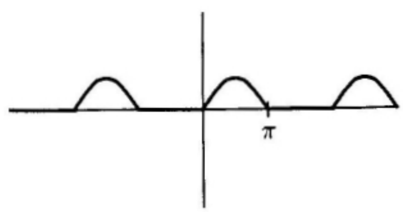

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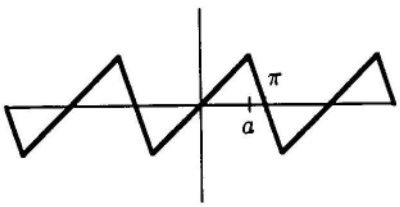
TABLE 1 (continued)

11.	$f(\theta) = \begin{cases} \theta & (-a < \theta < a) \\ a\frac{\pi-\theta}{\pi-a} & (a < \theta < \pi) \\ a\frac{\pi+\theta}{a-\pi} & (-\pi < \theta < -a) \end{cases}$	$\frac{2}{\pi - a}\sum_1^\infty \frac{\sin na}{n^2}\sin n\theta$
12.	$f(\theta) = \begin{cases} (2a)^{-1} & (\lvert\theta\rvert < a) \\ 0 & (a < \lvert\theta\rvert < \pi) \end{cases}$	$\frac{1}{2\pi} + \frac{1}{\pi}\sum_1^\infty \frac{\sin na}{na}\cos n\theta$
13.	$f(\theta) = \begin{cases} (2a)^{-1} & (\lvert\theta-\theta_0\rvert < a) \\ 0 & (a < \lvert\theta-\theta_0\rvert < \pi) \end{cases}$	$\frac{1}{2\pi} + \frac{1}{\pi}\sum_1^\infty \frac{\sin na}{na}(\cos n\theta_0 \cos n\theta$ $+ \sin n\theta_0 \sin n\theta)$
14.	$f(\theta) = \begin{cases} 1 & (-a < \theta < a) \\ -1 & (2a < \theta < 4a) \\ 0 & \text{elsewhere in } (-\pi, \pi) \end{cases}$	$\sum_1^\infty \frac{\sin na}{n}\big[(1 - \cos 3na)\cos n\theta$ $- \sin 3na \sin n\theta\big]$
15.	$f(\theta) = \begin{cases} a^{-2}(a-\lvert\theta\rvert) & (\lvert\theta\rvert < a) \\ 0 & (a < \lvert\theta\rvert < \pi) \end{cases}$	$\frac{1}{2\pi} + \frac{2}{\pi}\sum_1^\infty \frac{1 - \cos na}{n^2a^2}\cos n\theta$
16.	$f(\theta) = \theta^2 \quad (-\pi < \theta < \pi)$	$\frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos n\theta$
17.	$f(\theta) = \theta(\pi - \lvert\theta\rvert) \quad (-\pi < \theta < \pi)$	$\frac{8}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{(2n-1)^3}$
18.	$f(\theta) = e^{b\theta} \quad (-\pi < \theta < \pi)$	$\frac{\sinh b\pi}{\pi}\sum_{-\infty}^\infty \frac{(-1)^n}{b - in}e^{in\theta}$
19.	$f(\theta) = e^{b\theta} \quad (0 < \theta < 2\pi)$	$\frac{e^{2\pi b} - 1}{2\pi}\sum_{-\infty}^\infty \frac{e^{in\theta}}{b - in}$
20.	$f(\theta) = \sinh\theta \quad (-\pi < \theta < \pi)$	$\frac{2\sinh\pi}{\pi}\sum_1^\infty \frac{(-1)^{n+1}n}{n^2+1}\sin n\theta$



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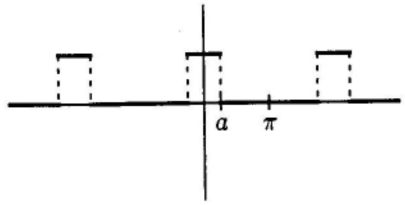

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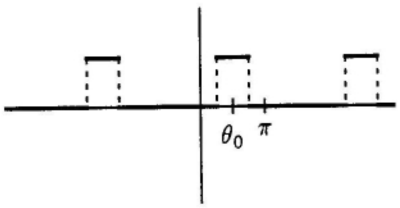

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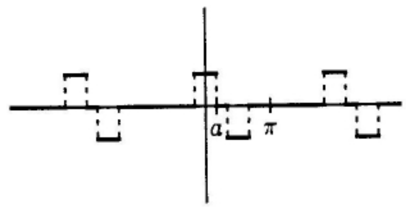

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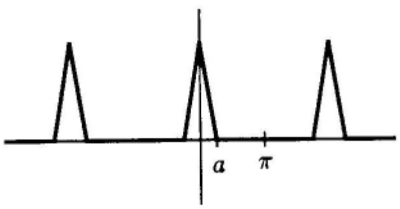

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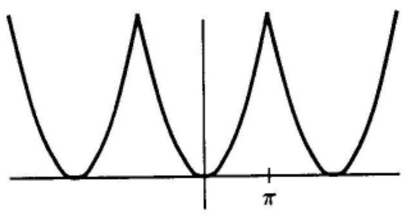

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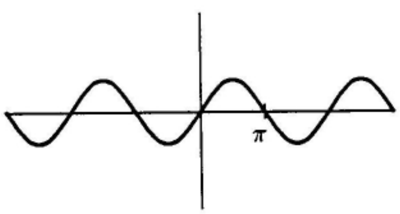

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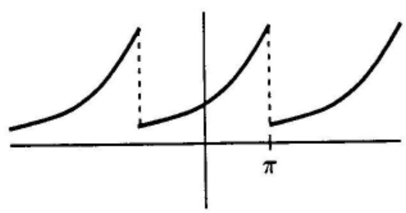

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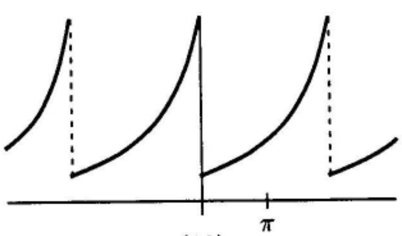

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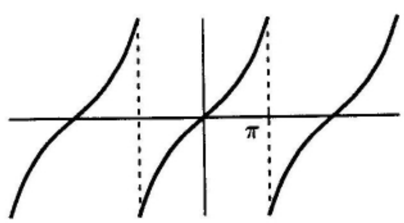


(20)

Bessel's Inequality. *If f is 2π-periodic and Riemann integrable on $[-\pi,\pi]$, and the Fourier coefficients c_n are defined by (2.5), then*

$$\sum_{-\infty}^{\infty} |c_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta.$$

Proof: Since $|z|^2 = z\overline{z}$ for any complex number z,

$$\begin{aligned}
&\left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta}\right|^2 \\
&= \left(f(\theta) - \sum_{-N}^{N} c_n e^{in\theta}\right)\left(\overline{f(\theta)} - \sum_{-N}^{N} \overline{c}_n e^{-in\theta}\right) \\
&= |f(\theta)|^2 - \sum_{-N}^{N}\left[c_n \overline{f(\theta)} e^{in\theta} + \overline{c}_n f(\theta) e^{-in\theta}\right] + \sum_{m,n=-N}^{N} c_m \overline{c}_n e^{i(m-n)\theta}.
\end{aligned}$$

Now divide both sides by 2π and integrate from $-\pi$ to π. Taking account of the formulas

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta = c_n, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)\theta}d\theta = \begin{cases} 0 & \text{if } m \ne n, \\ 1 & \text{if } m = n, \end{cases}$$

we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta}\right|^2 d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^{N}\left[c_n\overline{c}_n + \overline{c}_n c_n\right] + \sum_{-N}^{N} c_n \overline{c}_n \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^{N} |c_n|^2.
\end{aligned}$$

But the integral on the left is certainly nonnegative, so

$$0 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^{N} |c_n|^2.$$

Letting $N \to \infty$, we obtain the desired result. ∎

Bessel's inequality can also be stated in terms of the coefficients a_n and b_n defined by (2.6). Indeed, by equation (2.4), for $n \ge 1$ we have

$$\begin{aligned}
|a_n|^2 + |b_n|^2 &= a_n\overline{a}_n + b_n\overline{b}_n \\
&= (c_n + c_{-n})(\overline{c}_n + \overline{c}_{-n}) + i(c_n - c_{-n})(-i)(\overline{c}_n - \overline{c}_{-n}) \\
&= 2c_n\overline{c}_n + 2c_{-n}\overline{c}_{-n},
\end{aligned}$$

so that

$$|a_0|^2 = 4|c_0|^2, \qquad |a_n|^2 + |b_n|^2 = 2\left(|c_n|^2 + |c_{-n}|^2\right) \quad \text{for } n \geq 1.$$

Therefore,

$$\frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_1^\infty \left(|a_n|^2 + |b_n|^2\right) = \sum_{-\infty}^\infty |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta.$$

It turns out, as we shall see later, that Bessel's inequality is actually an equality. For now, its main significance is simply the fact that the series $\sum |a_n|^2$, $\sum |b_n|^2$, and $\sum |c_n|^2$ are all convergent. As a consequence, we obtain the following result, which is a special case of a theorem known as the *Riemann-Lebesgue lemma.*

Corollary 2.1. *The Fourier coefficients a_n, b_n, and c_n all tend to zero as $n \to \infty$ (and as $n \to -\infty$ in the case of c_n).*

Proof: $|a_n|^2$, $|b_n|^2$, and $|c_n|^2$ are the nth terms of convergent series, so they tend to zero as $n \to \infty$; hence so do a_n, b_n, and c_n. ∎

EXERCISES

Verify the formulas of Table 1. That is, for $3 \leq n \leq 20$, Exercise n is to show that the series in the right column of entry n in Table 1 is the Fourier series of the function in the left column. (Entries 1 and 2 are Examples 1 and 2 in the text.) Some of these functions are related to each other, and you may be able to use this fact to avoid caclulating all the Fourier coefficients from scratch each time. Entries 3 and 4 can be derived from entries 1 and 2; entry 7 can be derived from entry 6; entries 9 and 10 can be derived from entry 8; entries 13 and 14 can be derived from entry 12; and entries 19 and 20 can be derived from entry 18.

2.2 A convergence theorem

Question: does the Fourier series of a periodic function f converge to f? The answer is certainly not obvious — for example, why should one be able to expand nonsmooth functions like the examples in §2.1 in a series whose individual terms $\cos nx$ and $\sin nx$ possess derivatives of all orders? Fourier's assertion that the answer is yes was initially greeted with a certain amount of disbelief. In fact, the answer *is* always yes provided that things are interpreted suitably, although the situation is somewhat more delicate than one might initially expect.

In this section we shall show that the Fourier series of f converges to f under certain reasonably general hypotheses on f; later, in §2.3, §2.6, §3.4, and §9.3, we shall present some variants of this result under other conditions on f. We first define the class of functions with which we shall be working.

Suppose $-\infty < a < b < \infty$. We say that a function f on the closed interval $[a,b]$ is **piecewise continuous** provided that

(i) f is continuous on $[a,b]$ except perhaps at finitely many points $x_1,\dots,x_k$;

(ii) at each of the points $x_1,\dots,x_k$, the left-hand and right-hand limits of f,

$$f(x_j-) = \lim_{h\to 0,\ h>0} f(x_j - h) \quad \text{and} \quad f(x_j+) = \lim_{h\to 0,\ h>0} f(x_j + h),$$

exist. (If the endpoint a (or b) is one of the exceptional points x_j, we require only the right-hand (or left-hand) limit to exist.)

That is, f is piecewise continuous on $[a,b]$ if f is continuous there except for finitely many finite jump discontinuities. (When we say that the limits $f(x_j\pm)$ exist, we mean in particular that they are finite: ∞ is not allowed as a value.) We denote the class of piecewise continuous functions on $[a,b]$ by $PC(a,b)$.

Next, we say that a function f on $[a,b]$ is **piecewise smooth** if f and its first derivative f' are both piecewise continuous on $[a,b]$, and we denote the class of piecewise smooth functions on $[a,b]$ by $PS(a,b)$. More precisely, $f \in PS(a,b)$ if and only if

(i) $f \in PC(a,b)$;

(ii) f' exists and is continuous on (a,b) except perhaps at finitely many points $x_1,\dots,x_K$ (which will include any points where f is discontinuous), and the one-sided limits $f'(x_j-)$ and $f'(x_j+)$ $(j = 1,\dots,K)$, and also $f'(a+)$ and $f'(b-)$, exist.

In other words, f is piecewise smooth if its graph is a smooth curve except for finitely many jumps (where f is discontinuous) and corners (where f' is discontinuous). We do not allow infinite discontinuities (such as $f(x) = 1/x$ has at $x = 0$) or sharp cusps (where f' becomes infinite). See Figure 2.3.

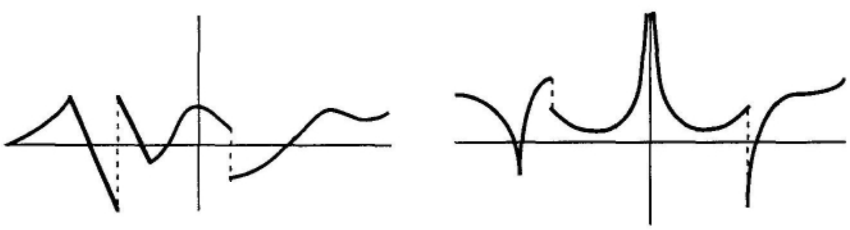


FIGURE 2.3. A piecewise smooth function (left) and a function that is not piecewise smooth (right).

One last bit of terminology: a function defined on the whole real line $\mathbf{R}$ is said to be **piecewise continuous** or **piecewise smooth** on $\mathbf{R}$ if it is so on every bounded interval $[a,b]$. (That is, f or f' may have infinitely many discontinuities on the whole line but only finitely many in any bounded interval.) We denote the spaces of piecewise continuous and piecewise smooth functions on $\mathbf{R}$ by $PC(\mathbf{R})$ and $PS(\mathbf{R})$.

We now return to consideration of the Fourier series of a 2π-periodic function $f(\theta)$. We recall that this is defined to be

$$\tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} \tag{2.9}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\cos n\psi \, d\psi, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\sin n\psi \, d\psi, \\ c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)e^{-in\psi}\, d\psi. \tag{2.10}$$

(We have labeled the variable of integration in (2.10) as ψ simply for later convenience.)

What meaning is to be attached to this series? Of course, the sum of any infinite series is defined to be the limit of its partial sums. When we write the series (2.9) in trigonometric form, we agree always to group together the terms involving $\cos n\theta$ and $\sin n\theta$ as indicated above; correspondingly, when we write it in exponential form, we agree always to group together the terms involving $e^{in\theta}$ and $e^{-in\theta}$. (*This convention will always be in force.*) Thus we take the Nth partial sum of the series (2.9) to be the sum $S_N^f(\theta)$ defined by

$$S_N^f(\theta) = \tfrac{1}{2}a_0 + \sum_1^N (a_n \cos n\theta + b_n \sin n\theta) = \sum_{-N}^{N} c_n e^{in\theta}, \tag{2.11}$$

and our aim is to show that S_N^f converges to f as $N \to \infty$.

If we plug the definition (2.10) of c_n into (2.11), we see that

$$S_N^f(\theta) = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\psi)e^{in(\theta-\psi)}d\psi = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\psi)e^{in(\psi-\theta)}d\psi.$$

The last equality results from replacing n by $-n$; this does not affect the sum since n ranges from $-N$ to N. If we now make the change of variable $\phi = \psi - \theta$ and use Lemma 2.1, we obtain

$$S_N^f(\theta) = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi+\theta}^{\pi+\theta} f(\theta+\phi)e^{in\phi}d\phi = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\theta+\phi)e^{in\phi}d\phi.$$

In short,

$$S_N^f(\theta) = \int_{-\pi}^{\pi} f(\theta+\phi)D_N(\phi)\,d\phi, \quad \text{where } D_N(\phi) = \frac{1}{2\pi}\sum_{-N}^{N} e^{in\phi}. \tag{2.12}$$

The function $D_N(\phi)$ is called the Nth **Dirichlet kernel**. We can express D_N in a more computable form by recognizing that it is the sum of a finite geometric progression:

$$D_N(\phi) = \frac{1}{2\pi}e^{-iN\phi}(1 + e^{i\phi} + \cdots + e^{i2N\phi}) = \frac{1}{2\pi}e^{-iN\phi}\sum_0^{2N} e^{in\phi}.$$

Since $\sum_0^K r^n = (r^{K+1} - 1)/(r - 1)$ for any $r \neq 1$, for $\phi \neq 0$ we have

$$D_N(\phi) = \frac{1}{2\pi}e^{-iN\phi}\frac{e^{i(2N+1)\phi} - 1}{e^{i\phi} - 1} = \frac{1}{2\pi}\frac{e^{i(N+1)\phi} - e^{-iN\phi}}{e^{i\phi} - 1}. \tag{2.13}$$

Moreover, on multiplying top and bottom by $e^{-i\phi/2}$, we obtain

$$D_N(\phi) = \frac{1}{2\pi}\frac{\exp\left[i(N + \frac{1}{2})\phi\right] - \exp\left[-i(N + \frac{1}{2})\phi\right]}{\exp(i\frac{1}{2}\phi) - \exp(-i\frac{1}{2}\phi)} = \frac{1}{2\pi}\frac{\sin(N + \frac{1}{2})\phi}{\sin\frac{1}{2}\phi}. \tag{2.14}$$

From this formula it is easy to sketch the graph of D_N: it is the rapidly oscillating sine wave $y = \sin(N + \frac{1}{2})\phi$ amplitude-modulated to fit inside the envelope $y = \pm(2\pi)^{-1}\csc\frac{1}{2}\phi$. See Figure 2.4.

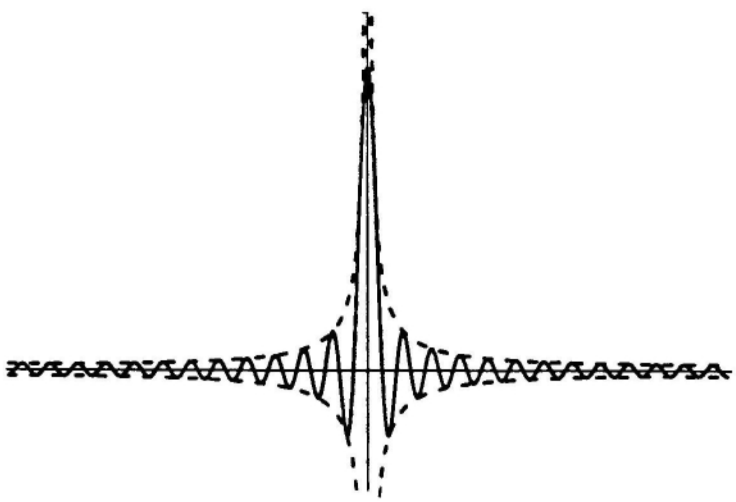


FIGURE 2.4. Graphs of the Dirichlet kernel $D_{25}(\phi)$ (solid) and its envelope $\pm(2\pi)^{-1}\csc\frac{1}{2}\phi$ (dashed) on the interval $-\pi < \phi < \pi$.

The pictorial intuition behind the fact that $S_N^f(\theta) \to f(\theta)$ is as follows: in the integral formula (2.12) for $S_N^f(\theta)$, the sharp central spike of $D_N(\phi)$ at $\phi = 0$ picks out the value $f(\theta)$, and the rapid oscillations of $D_N(\phi)$ away from $\phi = 0$ kill off most of the rest of the integral because of cancellations between positive and negative values. Before proceeding to the actual proof, however, we need one more fact about D_N.

Lemma 2.4. *For any N,*

$$\int_{-\pi}^{0} D_N(\theta)\,d\theta = \int_{0}^{\pi} D_N(\theta)\,d\theta = \frac{1}{2}.$$

Proof: From formula (2.12) we have

$$D_N(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{1}^{N} \cos n\theta,$$

so that

$$\int_{0}^{\pi} D_N(\theta)\,d\theta = \left[\frac{\theta}{2\pi} + \frac{1}{\pi}\sum_{1}^{N}\frac{\sin n\theta}{n}\right]_{0}^{\pi} = \frac{1}{2},$$

and likewise for the integral from $-\pi$ to 0. ∎

Here at last is our main convergence theorem. It says that the Fourier series of a function $f \in PS(\mathbf{R})$ converges pointwise to f, provided that we (re)define f at its points of discontinuity to be the average of its left- and right-hand limits.

Theorem 2.1. *If f is 2π-periodic and piecewise smooth on $\mathbf{R}$, and S_N^f is defined by (2.10) and (2.11), then*

$$\lim_{N\to\infty} S_N^f(\theta) = \tfrac{1}{2}\left[f(\theta-) + f(\theta+)\right]$$

for every θ. In particular, $\lim_{N\to\infty} S_N^f(\theta) = f(\theta)$ for every θ at which f is continuous.

Proof: By Lemma 2.4, we have

$$\tfrac{1}{2}f(\theta-) = f(\theta-)\int_{-\pi}^{0} D_N(\phi)\,d\phi, \qquad \tfrac{1}{2}f(\theta+) = f(\theta+)\int_{0}^{\pi} D_N(\phi)\,d\phi,$$

and hence by equation (2.12),

$$S_N^f(\theta) - \tfrac{1}{2}\left[f(\theta-) + f(\theta+)\right]$$
$$= \int_{-\pi}^{0}\left[f(\theta+\phi) - f(\theta-)\right]D_N(\phi)\,d\phi + \int_{0}^{\pi}\left[f(\theta+\phi) - f(\theta+)\right]D_N(\phi)\,d\phi.$$

We wish to show that for each fixed θ, this quantity approaches zero as $N \to \infty$. But by formula (2.13), we can write it as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)(e^{i(N+1)\phi} - e^{-iN\phi})\,d\phi \tag{2.15}$$

where $g(\phi)$ is defined to be

$$\frac{f(\theta+\phi)-f(\theta-)}{e^{i\phi}-1} \text{ for } -\pi<\phi<0, \qquad \frac{f(\theta+\phi)-f(\theta+)}{e^{i\phi}-1} \text{ for } 0<\phi<\pi.$$

g is a well-behaved function on $[-\pi,\pi]$, as smooth as f is, except near $\phi=0$ (where $e^{i\phi}-1$ vanishes). However, by l'Hôpital's rule,

$$\lim_{\phi\to 0+} g(\phi) = \lim_{\phi\to 0+} \frac{f(\theta+\phi)-f(\theta+)}{e^{i\phi}-1} = \lim_{\phi\to 0+} \frac{f'(\theta+\phi)}{ie^{i\phi}} = \frac{f'(\theta+)}{i}.$$

Similarly, $g(\phi)$ approaches the finite limit $i^{-1}f'(\theta-)$ as ϕ approaches zero from the left. Hence g is actually piecewise continuous on $[-\pi,\pi]$, so by the corollary to Bessel's inequality in §2.1, its Fourier coefficients

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)e^{-in\phi}\,d\phi$$

tend to zero as $n\to\pm\infty$. But the expression (2.15) is nothing but $C_{-(N+1)}-C_N$, so it vanishes as $N\to\infty$; and this is what we needed to show. ∎

Let us see what this theorem says with regard to the two examples of the previous section. The function f of Example 1 is piecewise smooth and everywhere continuous, so the Fourier series of f converges to f at every point. Thus,

$$\frac{\pi}{2}-\frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2} = |\theta| \quad \text{for } -\pi\le\theta\le\pi. \tag{2.16}$$

On the other hand, the function g of Example 2 is piecewise smooth and continuous except at the points $\theta=k\pi$ with k odd. At these discontinuities we have $g(k\pi-)=\pi$ and $g(k\pi+)=-\pi$, so $\frac{1}{2}[g(k\pi-)+g(k\pi+)]=0$. Thus the Fourier series of g converges to g at all points except $\theta=k\pi$ (k odd), where it converges to zero. Hence,

$$\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin n\theta = \frac{\theta}{2} \quad \text{for } -\pi<\theta<\pi. \tag{2.17}$$

In particular, if we take $\theta=0$ in (2.16), we obtain the formula

$$\sum_1^\infty \frac{1}{(2k-1)^2} = 1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots = \frac{\pi^2}{8}.$$

(As the reader may check, the same formula results from taking taking $\theta=\pi$.) Moreover, if we take $\theta=\frac{1}{2}\pi$ in (2.17) and use the fact that $\sin\frac{1}{2}n\pi$ is alternately 1 and -1 when n is odd and 0 when n is even, we find that

$$\sum_1^\infty \frac{(-1)^{k+1}}{2k-1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots = \frac{\pi}{4}.$$

These are two interesting instances where numerical series can be evaluated as special values of Fourier series. Others can be found in the exercises.

Theorem 2.1 says that the Fourier series of a 2π-periodic piecewise smooth function f converges to f everywhere, provided that f is (re)defined at each of its points of discontinuity to be the average of its left- and right-hand limits there. *Henceforth, when we speak of piecewise smooth functions, we shall assume that this adjustment has been made,* unless we explicitly state otherwise. This will obviate the need to single out the points of discontinuity for special attention. In particular, with this understanding, we have the following uniqueness theorem.

Corollary 2.2. *If f and g are 2π-periodic and piecewise smooth, and f and g have the same Fourier coefficients, then $f = g$.*

Proof: f and g are both the sum of the same Fourier series. ∎

EXERCISES

1. Which of the following functions are continuous, piecewise continuous, or piecewise smooth on $[-\pi, \pi]$?
   a. $f(\theta) = \csc\theta$. b. $f(\theta) = (\sin\theta)^{1/3}$. c. $f(\theta) = (\sin\theta)^{4/3}$.
   d. $f(\theta) = \cos\theta$ if $\theta > 0$, $f(\theta) = -\cos\theta$ if $\theta \leq 0$.
   e. $f(\theta) = \sin\theta$ if $\theta > 0$, $f(\theta) = \sin 2\theta$ if $\theta \leq 0$.
   f. $f(\theta) = (\sin\theta)^{1/5}$ if $\theta < \frac{1}{2}\pi$, $f(\theta) = \cos\theta$ if $\theta \geq \frac{1}{2}\pi$.
2. To what values do the series in entries 6, 7, 12, and 18 of Table 1, §2.1, converge at the points where their sums are discontinuous?

The Fourier series for a number of piecewise smooth functions are listed in Table 1 of §2.1, and Theorem 2.1 tells what the sums of these series are. By using this information and choosing suitable values of θ (usually 0, $\frac{1}{2}\pi$, or π), derive the following formulas for the sums of numerical series. (The relevant entries from Table 1 are indicated in parentheses.)

3. $\displaystyle\sum_1^\infty \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_1^\infty \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi-2}{4}$ (8).
4. $\displaystyle\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ (16).
5. $\displaystyle\sum_1^\infty \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$ (17).
6. $\displaystyle\sum_1^\infty \frac{(-1)^n}{n^2+b^2} = \frac{\pi}{2b}\operatorname{csch} b\pi - \frac{1}{2b^2}$ (18 or 19).
7. $\displaystyle\sum_1^\infty \frac{1}{n^2+b^2} = \frac{\pi}{2b}\coth b\pi - \frac{1}{2b^2}$ (18 or 19; this is a bit tricky).

2.3 Derivatives, integrals, and uniform convergence

This section is devoted to an examination of the behavior of Fourier series in relation to the processes of calculus.

We shall be largely concerned here with periodic functions that are both continuous and piecewise smooth. Pictorially, the graph of such a function is a smooth curve except that it can have "corners" where the derivative jumps. The fundamental theorem of calculus,

$$f(b) - f(a) = \int_a^b f'(\theta)\, d\theta,$$

applies to functions f that are continuous and piecewise smooth, even though f' is undefined at the "corners." To see this it suffices to express the integral on the right as the sum of integrals over the subintervals of $[a, b]$ on which f is differentiable; the continuity of f guarantees that the endpoint evaluations at the intermediate subdivision points cancel out. For example, if f is differentiable except at the point $c \in (a, b)$, we have

$$\begin{aligned}\int_a^b f'(\theta)\, d\theta &= \int_a^c f'(\theta)\, d\theta + \int_c^b f'(\theta) d\theta \\ &= \big[f(c) - f(a)\big] + \big[f(b) - f(c)\big] = f(b) - f(a).\end{aligned}$$

This observation will be used implicitly in several of the following calculations, including the proof of Theorem 2.2.

Our first main result relates the Fourier coefficients of a function to those of its derivative. The fact that this relation is so simple is one of the main reasons underlying the utility of Fourier series.

Theorem 2.2. *Suppose f is 2π-periodic, continuous, and piecewise smooth. Let a_n, b_n, and c_n be the Fourier coefficients of f defined in (2.5) and (2.6), and let a_n', b_n', and c_n' be the corresponding Fourier coefficients of f'. Then*

$$a_n' = nb_n, \qquad b_n' = -na_n, \qquad c_n' = inc_n.$$

Proof: This is a simple matter of integration by parts. For example,

$$c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\, d\theta = \frac{1}{2\pi} f(\theta)e^{-in\theta}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(-ine^{-in\theta})\, d\theta.$$

The first term on the right vanishes because $f(-\pi) = f(\pi)$ and $e^{in\pi} = e^{-in\pi} = (-1)^n$, and the second term is inc_n. The argument for a_n' and b_n' is the same; we leave the details to the reader. ∎

Combining this result with the theorem of the previous section, we easily obtain the basic results on differentiation and integration of Fourier series.

Theorem 2.3. *Suppose f is 2π-periodic, continuous, and piecewise smooth, and suppose also that f' is piecewise smooth. If*

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} = \tfrac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

is the Fourier series of $f(\theta)$, then $f'(\theta)$ is the sum of the derived series

$$\sum_{-\infty}^{\infty} inc_n e^{in\theta} = \sum_{1}^{\infty}(nb_n \cos n\theta - na_n \sin n\theta)$$

for all θ at which $f'(\theta)$ exists. At the exceptional points where f' has jumps, the series converges to $\frac{1}{2}\left[f'(\theta-) + f'(\theta+)\right]$.

Proof: Since f' is piecewise smooth, by Theorem 2.1 it is the sum of its Fourier series at every point (with appropriate modifications at the jumps). By Theorem 2.2, the coefficients of $e^{in\theta}$, $\cos n\theta$, and $\sin n\theta$ in this series are inc_n, nb_n, and $-na_n$. The result follows. ∎

In considering integration of Fourier series, one must keep in mind that the indefinite integral of a periodic function may not be periodic. For example, the constant function $f(\theta) = 1$ is periodic, but its antiderivative $F(\theta) = \theta$ is not. However, the integral of every term in a Fourier series is periodic except for the constant term, from which we see that a periodic function has a periodic integral precisely when the constant term in its Fourier series vanishes, i.e., when its mean value on $[-\pi, \pi]$ is zero. We therefore arrive at the following result.

Theorem 2.4. *Suppose f is 2π-periodic and piecewise continuous, with Fourier coefficients a_n, b_n, c_n, and let $F(\theta) = \int_0^\theta f(\phi)\,d\phi$. If $c_0 \,(= \frac{1}{2}a_0) = 0$, then for all θ we have*

$$F(\theta) = C_0 + \sum_{n\neq 0} \frac{c_n}{in} e^{in\theta} = \tfrac{1}{2}A_0 + \sum_{1}^{\infty}\left(\frac{a_n}{n}\sin n\theta - \frac{b_n}{n}\cos n\theta\right) \tag{2.18}$$

where the constant term is the mean value of F on $[-\pi, \pi]$:

$$C_0 = \tfrac{1}{2}A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)\,d\theta. \tag{2.19}$$

The series on the right of (2.18) is the series obtained by formally integrating the Fourier series of f term by term, whether the latter series actually converges or not. If $c_0 \neq 0$, the sum of the series on the right of (2.18) is $F(\theta) - c_0\theta$.

Proof: F is continuous and piecewise smooth, being the integral of a piecewise continuous function. Moreover, if $c_0 = 0$, F is 2π-periodic, for

$$F(\theta + 2\pi) - F(\theta) = \int_\theta^{\theta+2\pi} f(\phi)\,d\phi = \int_{-\pi}^{\pi} f(\phi)\,d\phi = 2\pi c_0 = 0.$$

Hence, by Theorem 2.1, $F(\theta)$ is the sum of its Fourier series at every θ. But by Theorem 2.2 applied to F, the Fourier coefficients A_n, B_n, and C_n of F are related to those of f by

$$A_n = -\frac{b_n}{n}, \qquad B_n = \frac{a_n}{n}, \qquad C_n = \frac{c_n}{in} \qquad (n \neq 0).$$

The formula (2.19) for the constant C_0 or $\frac{1}{2}A_0$ is just the usual formula for the zeroth Fourier coefficient of F. If $c_0 \neq 0$, these arguments can be applied to the function $f(\theta) - c_0$ rather than $f(\theta)$, yielding the final assertion. ∎

Example. Let f be the periodic function such that $f(\theta) = 1$ for $0 < \theta < \pi$ and $f(\theta) = -1$ for $-\pi < \theta < 0$, and let $F(\theta) = \int_0^\theta f(\phi)\,d\phi$. Clearly $F(\theta) = |\theta|$ for $|\theta| \leq \pi$. By entry 4 of Table 1, §2.1, the Fourier series of f is $(4/\pi)\sum_1^\infty (2n-1)^{-1}\sin(2n-1)\theta$, so by Theorem 2.4 we have

$$F(\theta) = C_0 - \frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2} \quad \text{where } C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta|\,d\theta = \frac{\pi}{2}.$$

Thus we recover the result of entry 2 of Table 1.

Theorem 2.1 gave conditions under which the Fourier series of f converges pointwise to f. However, experience in working with infinite series teaches us that simple pointwise convergence of a series can be a tricky business, and that we are much better off if the convergence is absolute and uniform. We recall the definitions: suppose the series $\sum_1^\infty g_n(x)$ converges to $g(x)$ on a set S. The convergence is *absolute* if the series $\sum_1^\infty |g_n(x)|$ also converges for $x \in S$, and *uniform* if not only does the difference $g(x) - \sum_1^N g_n(x)$ tend to zero for each $x \in S$, but so does the maximum of this difference over the whole set S:

$$\sup_{x \in S}\left| g(x) - \sum_1^N g_n(x) \right| \to 0 \quad \text{as } N \to \infty.$$

The most useful criterion for guaranteeing absolute and uniform convergence is the **Weierstrass M-test**: if there is a sequence M_n of positive constants such that

$$|g_n(x)| \leq M_n \quad \text{for } x \in S, \quad \text{and} \quad \sum_1^\infty M_n < \infty,$$

then the series $\sum_1^\infty g_n(x)$ is absolutely and uniformly convergent.

In the case of Fourier series, we have the obvious estimates

$$|a_n \cos n\theta| \leq |a_n|, \qquad |b_n \sin n\theta| \leq |b_n|, \qquad |c_n e^{in\theta}| = |c_n|.$$

Hence the Weierstrass M-test will apply to a Fourier series in trigonometric form if $\sum_0^\infty |a_n| < \infty$ and $\sum_1^\infty |b_n| < \infty$, and to a Fourier series in exponential form if

$\sum_{-\infty}^{\infty} |c_n| < \infty$. Since it follows from the equations (2.3) and (2.4) relating a_n, b_n, and c_n that

$$|c_{\pm n}| \le |a_n| + |b_n|, \qquad |a_n| \le |c_n| + |c_{-n}|, \qquad |b_n| \le |c_n| + |c_{-n}|,$$

the conditions $\sum_0^{\infty} |a_n| < \infty$ and $\sum_1^{\infty} |b_n| < \infty$ are completely equivalent to the condition $\sum_{-\infty}^{\infty} |c_n| < \infty$. We now present a sufficient (but not necessary) condition for them to hold.

Theorem 2.5. *If f is 2π-periodic, continuous, and piecewise smooth, then the Fourier series of f converges to f absolutely and uniformly on* $\mathbf{R}$.

Proof: By Theorem 2.1 and the remarks just made, it suffices to show that the series $\sum_{-\infty}^{\infty} |c_n|$ converges. Let c_n' denote the Fourier coefficients of f'. By Theorem 2.2 we know that $c_n = (in)^{-1} c_n'$ for $n \neq 0$, and by Bessel's inequality applied to f',

$$\sum_{-\infty}^{\infty} |c_n'|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(\theta)|^2 d\theta < \infty.$$

Hence, by the Cauchy-Schwarz inequality,

$$\sum_{-\infty}^{\infty} |c_n| = |c_0| + \sum_{n\neq 0} \left|\frac{c_n'}{n}\right| \le |c_0| + \left(\sum_{n\neq 0} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n\neq 0} |c_n'|^2\right)^{1/2} < \infty,$$

since $\sum_{n\neq 0}(1/n^2) = 2\sum_1^{\infty}(1/n^2) < \infty$. (In case the reader needs reminding: the Cauchy-Schwarz inequality says that the dot product of two vectors is bounded by the product of their norms. It is valid in any number n of dimensions and also in the limit as $n \to \infty$. We shall discuss it in more detail in Chapter 3.) ∎

Let us return to Theorem 2.3. If f has many derivatives, Theorem 2.3 can be applied several times in succession to calculate the Fourier series of f', f'', f''', etc. Each time one takes a derivative, the magnitude of the Fourier coefficients c_n (or a_n and b_n) increases by a factor of $|n|$, which means that the derived series converges more slowly than the original series. Or, to put it another way, if the derived series converges at all, the original series must converge relatively rapidly. Thus there is a connection between the differentiability properties of a function and the rate of convergence of its Fourier series. Here is a precise result along these lines.

Theorem 2.6. *Suppose f is 2π-periodic. If f is of class $C^{(k-1)}$ and $f^{(k-1)}$ is piecewise smooth (thus $f^{(k)}$ exists except at finitely many points in each bounded interval and is piecewise continuous), then the Fourier coefficients of f satisfy*

$$\sum |n^k a_n|^2 < \infty, \qquad \sum |n^k b_n|^2 < \infty, \qquad \sum |n^k c_n|^2 < \infty.$$

In particular,

$$n^k a_n \to 0, \qquad n^k b_n \to 0, \qquad n^k c_n \to 0 \qquad \text{as } n \to \infty.$$

On the other hand, suppose the Fourier coefficients c_n $(n \neq 0)$ satisfy $|c_n| \le C|n|^{-(k+\alpha)}$ (equivalently, $|a_n| \le Cn^{-(k+\alpha)}$ and $|b_n| \le Cn^{-(k+\alpha)}$) for some $C > 0$ and $\alpha > 1$. Then f is of class $C^{(k)}$.

Proof: For the first part, we apply Theorem 2.2 k times to conclude that the Fourier coefficients $c_n^{(k)}$ of $f^{(k)}$ are given by $c_n^{(k)} = (in)^k c_n$, and similarly for $a_n^{(k)}$ and $b_n^{(k)}$. The conclusions then follow from Bessel's inequality (applied to $f^{(k)}$) and its corollary. For the second part, we observe that since $\alpha > 1$,

$$\sum_{n\neq 0} |n^j c_n| \leq C \sum_{n\neq 0} |n|^{-(k-j+\alpha)} \leq 2C \sum_{n>0} n^{-\alpha} < \infty \quad \text{for } j \leq k.$$

Thus, by the Weierstrass M-test, the series $\sum_{-\infty}^{\infty}(in)^j c_n e^{in\theta}$ are absolutely and uniformly convergent for $j \leq k$. They therefore define continuous functions, which are the derivatives $f^{(j)}$ of $f(\theta) = \sum c_n e^{in\theta}$. ∎

The two halves of Theorem 2.6 are not perfect converses of each other; this is in the nature of things. (There is no simple "if and only if" theorem of this sort.) However, the moral is clear: the more derivatives a function has, the more rapidly its Fourier coefficients will tend to zero, and vice versa. In particular, f has derivatives of all orders precisely when its Fourier coefficients tend to zero more rapidly than any power of n (for example, $c_n = e^{-\epsilon|n|}$).

Another aspect of this phenomenon: the basic functions $e^{in\theta}$ or $\cos n\theta$ and $\sin n\theta$ are, of course, perfectly smooth individually, but they become more "jagged," that is, more highly oscillatory, as $n \to \infty$. In order to synthesize non-smooth functions from these smooth ingredients, then, the proper technique is to use relatively large amounts of the high-frequency (i.e., large-n) functions.

These points are worth remembering; they are among the basic lessons of Fourier analysis. The reader can see how they work by examining the entries Table 1 in §2.1. For instance, the sawtooth wave in entry 2 is piecewise smooth but not continuous; its Fourier coefficients are on the order of n^{-1}. The triangle wave in entry 1 is one step better — continuous and piecewise smooth, with a piecewise smooth derivative; its Fourier coefficients are on the order of n^{-2}. These examples are quite typical.

EXERCISES

1. Derive the result of entry 16 of Table 1, §2.1, by using equation (2.17) and Theorem 2.4.
2. Starting from entry 16 of Table 1 and using Theorem 2.4, show that

   a. $\theta^3 - \pi^2\theta = 12\sum_1^\infty \dfrac{(-1)^n \sin n\theta}{n^3} \qquad (-\pi \leq \theta \leq \pi);$

   b. $\theta^4 - 2\pi^2\theta^2 = 48\sum_1^\infty \dfrac{(-1)^{n+1}\cos n\theta}{n^4} - \dfrac{7\pi^4}{15} \qquad (-\pi \leq \theta \leq \pi);$

   c. $\sum_1^\infty \dfrac{1}{n^4} = \dfrac{\pi^4}{90}.$

3. Evaluate $\sum_1^\infty (2n-1)^{-4}\cos(2n-1)\theta$ by using entry 17 of Table 1.

4. By entry 8 of Table 1, we have

$$\sin\theta = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2-1} \qquad (0 \le \theta \le \pi), \qquad (*)$$

and we also have

$$\cos\theta = \frac{d}{d\theta}\sin\theta = -\int_{\pi/2}^{\theta} \sin\phi\, d\phi.$$

Show that the series $(*)$ can be differentiated and integrated termwise to yield two apparently different expressions for $\cos\theta$ for $0 < \theta < \pi$, and reconcile these two expressions. (Hint: Equation (2.17) is useful.)

5. Let $f(\theta)$ be the periodic function such that $f(\theta) = e^\theta$ for $-\pi < \theta \le \pi$, and let $\sum_{-\infty}^{\infty} c_n e^{in\theta}$ be its Fourier series; thus $e^\theta = \sum c_n e^{in\theta}$ for $|\theta| < \pi$. If we formally differentiate this equation, we obtain $e^\theta = \sum inc_n e^{in\theta}$. But then $c_n = inc_n$, or $(1 - in)c_n = 0$, so $c_n = 0$ for all n. This is obviously wrong; where is the mistake?

6. The Fourier series in entries 11 and 12 of Table 1 are clearly related: the second is close to being the derivative of the first. Find the exact relationship (a) by examining the series and (b) by examining the functions that the series represent.

7. How smooth are the following functions? That is, how many derivatives can you guarantee them to have?

a. $f(\theta) = \sum_{-\infty}^{\infty} \frac{e^{in\theta}}{n^{13.2} + 2n^6 - 1}$. b. $f(\theta) = \sum_0^\infty \frac{\cos n\theta}{2^n}$.

c. $f(\theta) = \sum_0^\infty \frac{\cos 2^n\theta}{2^n}$.

2.4 Fourier series on intervals

Fourier series give expansions of periodic functions on the line in terms of trigonometric functions. They can also be used to give expansions of functions defined on a finite interval in terms of trigonometric functions on that interval.

Suppose the interval in question is $[-\pi, \pi]$. (Other intervals can be transformed into this one by a linear change of variable; we shall discuss this point later.) Given a bounded, integrable function f on $[-\pi, \pi]$, we extend it to the whole real line by requiring it to be periodic of period 2π. Actually, to be completely consistent about this we should start out with f defined only on the half-open interval $(-\pi, \pi]$ or $[-\pi, \pi)$, or else (re)define f at the endpoints so that

$f(-\pi) = f(\pi)$. To be definite, we follow the first course of action; then the **periodic extension** of f to the whole line is given by

$$f(\theta + 2n\pi) = f(\theta) \quad \text{for all } \theta \in (-\pi, \pi] \text{ and all integers } n.$$

For instance, the periodic functions discussed in Examples 1 and 2 of §2.1 are the periodic extiensions of the functions $f(\theta) = |\theta|$ and $g(\theta) = \theta$ from $(-\pi, \pi]$ to the whole line.

If f is a piecewise smooth function on $(-\pi, \pi]$, we can expand its periodic extension in a Fourier series, and then by restricting the variable θ to $[-\pi, \pi]$, we obtain an expansion of the original function. All of what we have said in the previous sections applies to this situation, but there is one point that needs attention. If the original f is piecewise continuous or piecewise smooth on $[-\pi, \pi]$, then its periodic extension will be piecewise continuous or piecewise smooth on $\mathbf{R}$. However, even if f is perfectly smooth on $[-\pi, \pi]$, there will generally be discontinuities in the extended function or its derivatives at the points $(2n+1)\pi$, n an integer, where (so to speak) the copies of f are glued together. To be precise, suppose f is continuous on $[-\pi, \pi]$. Then the extension will be continuous at the points $(2n+1)\pi$ if and only if $f(-\pi) = f(\pi)$, and in this case the extension will have derivatives up to order k at $(2n+1)\pi$ if and only if $f^{(j)}(-\pi+) = f^{(j)}(\pi-)$ for $j \leq k$. (This is illustrated by the examples in §2.1: see Figures 2.1(a) and 2.2(a).) These phenomena must be kept in mind when one studies the relations between the smoothness properties of f and the size of its Fourier coefficients as in Theorem 2.6.

Two interesting variations can be made on this theme. Suppose now that we are interested in functions on the interval $[0, \pi]$ rather than $[-\pi, \pi]$. We can make such a function f into a 2π-periodic function, and hence obtain a Fourier expansions for it, by a twofold extension process: first we extend f in some simple way to the interval $[-\pi, \pi]$, then we extend the result periodically. There are two standard ways of performing the first step: we extend f to $[-\pi, \pi]$ by declaring it to be either even or odd. That is, we have the **even extension** f_{even} of f to $[-\pi, \pi]$ defined by

$$f_{\text{even}}(-\theta) = f(\theta) \text{ for } \theta \in [0, \pi]$$

and the **odd extension** f_{odd} of f to $[-\pi, \pi]$ defined by

$$f_{\text{odd}}(-\theta) = -f(\theta) \text{ for } \theta \in (0, \pi], \qquad f_{\text{odd}}(0) = 0.$$

(See Figure 2.5.) The advantage of using f_{even} or f_{odd} rather than any other extension is that the Fourier coefficients turn out very simply. Indeed, it follows from Lemma 2.2 of §2.1 that

$$\int_{-\pi}^{\pi} f_{\text{even}}(\theta) \cos n\theta \, d\theta = 2\int_0^{\pi} f(\theta) \cos n\theta \, d\theta, \qquad \int_{-\pi}^{\pi} f_{\text{even}}(\theta) \sin n\theta \, d\theta = 0,$$

whereas

$$\int_{-\pi}^{\pi} f_{\text{odd}}(\theta)\cos n\theta\, d\theta = 0, \qquad \int_{-\pi}^{\pi} f_{\text{odd}}(\theta)\sin n\theta\, d\theta = 2\int_0^{\pi} f(\theta)\sin n\theta\, d\theta.$$

Thus the Fourier series of f_{even} involves only cosines and the Fourier series of f_{odd} involves only sines; moreover, the Fourier coefficients for these two cases can be computed in terms of the values of the original function f on $[0, \pi]$. We are thus led to the following definitions.

Definition. Suppose f is an integrable function on $[0, \pi]$. The series

$$\tfrac{1}{2}a_0 + \sum_1^{\infty} a_n \cos n\theta, \quad \text{where } a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta,$$

is called the **Fourier cosine series** of f. The series

$$\sum_1^{\infty} b_n \sin n\theta, \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta,$$

is called the **Fourier sine series** of f.

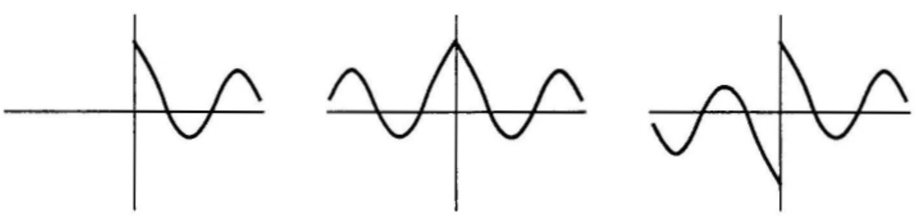


FIGURE 2.5. A function defined on $[0, \pi]$ (left), its even extension (middle), and its odd extension (right).

If f is piecewise continuous or piecewise smooth on $[0, \pi]$, its even periodic and odd periodic extensions will have the same properties on **R**, but as before, one must watch for extra discontinuities at the points $n\pi$ (n an integer) where the pieces are joined together. If f is continuous on $[0, \pi]$, the even periodic extension will be continuous everywhere, but its derivative will have jumps at the points $2n\pi$ or $(2n+1)\pi$ unless $f'(0+) = 0$ or $f'(\pi-) = 0$, respectively. The odd periodic extension is less forgiving: it will have discontinuities at the points $2n\pi$ or $(2n+1)\pi$ unless $f(0) = 0$ or $f(\pi) = 0$, respectively. (As for higher derivatives: there are potential problems with the odd-order derivatives of the even periodic extension and with the even-order derivatives of the odd periodic extension at the points $n\pi$.)

Example 1. Consider the function $f(\theta) = \theta$ on $[0, \pi]$. Its even and odd periodic extensions are given on $(-\pi, \pi)$ by $f_{\text{even}}(\theta) = |\theta|$ and $f_{\text{odd}}(\theta) = \theta$; these are the functions whose Fourier series we worked out in §2.1. Hence,

$$\theta = 2\sum_1^\infty \frac{(-1)^{n+1}\sin n\theta}{n} = \frac{\pi}{2} - \frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2} \qquad (0 < \theta < \pi).$$

Here f is perfectly smooth on $[0, \pi]$, but f_{odd} has discontinuities at the odd multiples of π. f_{even} is continuous everywhere, but its first derivative has discontinuities at all integer multiples of π. The reader may find other examples in Table 1.

At any rate, if we keep these remarks in mind and apply Theorem 2.1, we arrive at the following result.

Theorem 2.7. *Suppose f is piecewise smooth on $[0, \pi]$. The Fourier cosine series and the Fourier sine series of f converge to $\frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$ at every $\theta \in (0, \pi)$. In particular, they converge to $f(\theta)$ at every $\theta \in (0, \pi)$ where f is continuous. The Fourier cosine series of f converges to $f(0+)$ at $\theta = 0$ and to $f(\pi-)$ at $\theta = \pi$; the Fourier sine series of f converges to 0 at both these points.*

The results of the previous section on termwise differentiation and uniform convergence can be applied to these series, provided that one takes account of the behavior at the endpoints as indicated above.

Finally, we may wish to consider periodic functions whose period is something other than 2π, or functions defined on intervals other than $[-\pi, \pi]$ or $[0, \pi]$. These situations can be reduced to the ones we have already studied by making a simple change of variable.

For instance, suppose $f(x)$ is a periodic function with period $2l$. (The factor of 2 is merely for convenience.) We make the change of variables

$$x = \frac{l\theta}{\pi}, \qquad g(\theta) = f(x) = f\left(\frac{l\theta}{\pi}\right).$$

Then g is 2π-periodic, so if it is piecewise smooth we can expand it in a Fourier series:

$$g(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-in\theta}\,d\theta.$$

If we now substitute $\theta = \pi x/l$ into these formulas, we obtain the $2l$-periodic Fourier series of the original function f:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}, \qquad c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx. \tag{2.20}$$

The corresponding formula in terms of cosines and sines is

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty \left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right], \tag{2.21}$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx. \tag{2.22}$$

From this it follows that the Fourier cosine and sine expansions of a piecewise smooth function f on the interval $[0, l]$ are

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx, \tag{2.23}$$

and

$$f(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \tag{2.24}$$

These formulas are probably worth memorizing; they are used very frequently. Another point worth remembering is that, just as in the case of Fourier series for periodic functions, *the constant term* $\frac{1}{2}a_0$ *in the Fourier cosine series of a function* f *on an interval is the mean value of* f *on that interval:* $\frac{1}{2}a_0 = l^{-1}\int_0^l f(x)\,dx$.

Example 2. Let us find the Fourier cosine and sine expansions of $f(x) = x$ on $[0, l]$. Having set $\theta = \pi x/l$, this amounts to finding the expansions of $g(\theta) = l\theta/\pi$ on $[0, \pi]$, which we have done above. Namely, for $0 < \theta < \pi$ we have

$$\frac{l\theta}{\pi} = \frac{2l}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin n\theta = \frac{l}{2} - \frac{4l}{\pi^2}\sum_1^\infty \frac{1}{(2n-1)^2}\cos(2n-1)\theta,$$

so for $0 < x < l$,

$$x = \frac{2l}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l} = \frac{l}{2} - \frac{4l}{\pi^2}\sum_1^\infty \frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}.$$

Finally, what if we wish to use an interval of length l whose left endpoint is not 0, say $[a, a+l]$? Simply apply the preceding formulas to $g(x) = f(x+a)$; we leave it to the reader to write out the resulting formulas for $f(x)$.

EXERCISES

In Exercises 1–6, find both the Fourier cosine series and the Fourier sine series of the given function on the interval $[0, \pi]$. Try to use the results of Table 1, §2.1, rather than working from scratch. To what values do these series converge when $\theta = 0$ and $\theta = \pi$?

1. $f(\theta) = 1$.
2. $f(\theta) = \pi - \theta$.
3. $f(\theta) = \sin\theta$.
4. $f(\theta) = \cos\theta$.

5. $f(\theta) = \theta^2$. (For the sine series, use entries 1 and 17 of Table 1.)
6. $f(\theta) = \theta$ for $0 \le \theta \le \frac{1}{2}\pi$, $f(\theta) = \pi - \theta$ for $\frac{1}{2}\pi \le \theta \le \pi$. (For the sine series, use entry 11 of Table 1, and for the cosine series, entry 2.)

In Exercises 7–11, expand the function in a series of the indicated type. For example, "sine series on $[0, l]$" means a series of the form $\sum b_n \sin(n\pi x/l)$. Again, use previously derived results as much as possible.

7. $f(x) = 1$; sine series on $[0, 6\pi]$.
8. $f(x) = 1 - x$; cosine series on $[0, 1]$.
9. $f(x) = 1$ for $0 < x < 2$, $f(x) = -1$ for $2 < x < 4$; cosine series on $[0, 4]$.
10. $f(x) = lx - x^2$; sine series on $[0, l]$.
11. $f(x) = e^x$; series of the form $\sum_{-\infty}^{\infty} c_n e^{2\pi i n x}$ on $[0, 1]$.
12. Suppose f is a piecewise continuous function on $[0, \pi]$ such that $f(\theta) = f(\pi - \theta)$. (That is, the graph of f is symmetric about the line $\theta = \frac{1}{2}\pi$.) Let a_n and b_n be the Fourier cosine and sine coefficients of f. Show that $a_n = 0$ for n odd and $b_n = 0$ for n even.

2.5 Some applications

At this point we are ready to complete the solutions of the boundary value problems that were discussed in §1.3. The first of these problems was the one describing heat flow on an interval $[0, l]$, where the initial temperature is $f(x)$ and the endpoints are held at temperature zero,

$$u_t = ku_{xx}, \qquad u(x,0) = f(x) \quad \text{for } x \in [0, l], \qquad u(0,t) = u(l,t) = 0 \quad \text{for } t > 0,$$

and we derived the following series as a candidate for a solution:

$$\begin{aligned} u(x,t) &= \sum_1^\infty b_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \sin\frac{n\pi x}{l}, \\ \text{where} \quad f(x) &= \sum_1^\infty b_n \sin\frac{n\pi x}{l}. \end{aligned} \tag{2.25}$$

The questions that we left open were: (1) Can the initial temperature f be expressed as such a sine series? (2) Does this formula for u actually define a solution of the heat equation with the given boundary conditions? We now know that the answer to the first question is *yes*, provided that f is piecewise smooth on $[0, l]$ (certainly a reasonable requirement from a physical point of view): we have merely to expand f in its Fourier sine series (2.24). Let us therefore address the second question.

The individual terms in the series for u solve the heat equation, by the way they were constructed. Moreover, when $t > 0$ the factor $\exp(-n^2\pi^2 kt/l)$ tends to zero very rapidly as $n \to \infty$, so that the series converges nicely. More precisely, since the coefficients b_n tend to zero as $n \to \infty$ and in particular are bounded by some constant C, for any positive ϵ we have

$$0 < \left| b_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \sin \frac{n\pi x}{l} \right| \le Ce^{-\delta n^2} \quad \text{for } t \ge \epsilon, \text{ where } \delta = \frac{\pi^2 k\epsilon}{l^2}.$$

The same sort of estimate also holds for the first t-derivative and the first two x-derivatives of the terms of the series for u, with an extra factor of n^2 thrown in. Since $\sum_1^\infty n^k e^{-\delta n^2}$ converges for any k, we see by the Weierstrass M-test that these derived series converge absolutely and uniformly in the region $0 \le x \le l$, $t \ge \epsilon$, and we deduce that termwise differentiation of the series is permissible. Conclusion: u is a solution of the heat equation.

As for the boundary conditions, it is evident that $u(0,t) = u(l,t) = 0$, since all the terms in the series for u vanish at $x = 0, l$, and $u(x,0) = f(x)$ by the choice of the coefficients b_n. However, as we pointed out in §1.1, we really want a bit more, namely, the continuity condition that $u(x,t)$ should tend to zero as $x \to 0, l$ and to $f(x)$ as $t \to 0$. The preceding discussion shows that the first of these requirements is always satisfied: for each $t > 0$, the series for $u(x,t)$ converges uniformly on $[0,l]$,so $u(x,t)$ is a continuous function of x. (In particular, as $x \to 0$ or $x \to l$, $u(x,t)$ approaches $u(0,t)$ or $u(l,t)$, which are zero.) Moreover, if f is continuous and piecewise smooth on $[0,l]$ and $f(0) = f(l) = 0$, then the odd periodic extension of f is continuous and piecewise smooth, so $\sum |b_n| < \infty$ by Theorem 2.5. The Weierstrass M-test then shows that the series for u converges uniformly on the whole region $0 \le x \le l$, $t \ge 0$, and hence that u is continuous there; in particular, $u(x,t) \to u(x,0) = f(x)$ as $t \to 0$.

If f has discontinuities or is nonzero at the endpoints, it is still true that $u(x,t) \to f(x)$ as $t \to 0$ provided that $0 < x < l$ and f is continuous at x, but the proof is more delicate. (See Walker [53], §4.7.) We shall not concern ourselves with such technical refinements, as we have already established the main point: under reasonable assumptions on the initial temperature f, the function u satisfies all the desired conditions.

One question we have not really settled is the uniqueness of the solution. That is, we have constructed *one* solution; is it the only one? The answer is *yes*. One can argue that any solution $u(x,t)$ must be expandable in a Fourier sine series in x for each t and then use the differential equation to show that the coefficients of this series must be the ones we found above. Alternatively, one can invoke some general uniqueness theorems for solutions of the heat equation; see John [33] or Folland [24]. Similar considerations apply to the other problems we solve later, and we shall not worry about uniqueness from now on except in situations where pitfalls actually exist.

Lest the reader become too complacent, however, let us briefly consider the problem of solving the heat equation for times $t < 0$ — that is, given the temperature distribution at time $t = 0$, to reconstruct the distribution at earlier times. If we take $t < 0$ in (2.25), the factors $e^{-n^2\pi^2 kt/l^2}$ tend rapidly to *infinity* rather than zero as $n \to \infty$, with the result that the series for $u(x,t)$ will almost certainly *diverge* unless the coefficients b_n of the initial function f happen to decay extremely rapidly as $n \to \infty$. Thus (2.25), in general, does *not* give a solution to the heat equation when $t < 0$. This is not merely a failure of mathematical technique, however. The initial value problem for the time-reversed heat equation is simply not well posed, a reflection of the fundamental physical fact that the direction of time is irreversible in diffusion processes. One can mix hot water and cold water to get warm water, but one cannot then separate the warm water back into hot and cold components! More to the point, one cannot tell by examining the warm water which part was initially hot and which part was initially cold, or what their initial temperatures were.

Exactly the same considerations apply to the problem of heat flow on $[0, l]$ with insulated endpoints,

$$u_t = ku_{xx}, \qquad u(x,0) = f(x), \qquad u_x(0,t) = u_x(l,t) = 0,$$

whose solution is

$$u(x,t) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right)\cos\frac{n\pi x}{l},$$

where

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos\frac{n\pi x}{l}.$$

The only difference is that now we expand f in its Fourier cosine series (2.22).

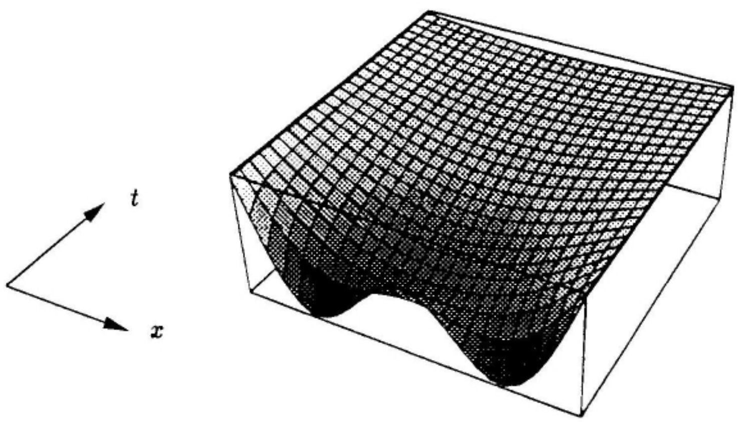


FIGURE 2.6. The solution (2.25) of the heat equation with $k = \frac{1}{4}$, $l = 1$, $b_1 = -\frac{1}{3}$, $b_2 = -\frac{1}{6}$, and $b_n = 0$ for $n > 2$, on the region $0 \le x \le 1$, $0 \le t \le 1$.

Let us pause a moment to see what these solutions tell us about the physics of the situation. In the limit as $t \to \infty$, the exponential factors all vanish, so the solution u approaches a constant — namely, 0 in the case where the endpoints are held at temperature 0 and $\frac{1}{2}a_0$ in the case of insulated endpoints. The first of these is easy to understand: the interval $[0, l]$ comes into thermal equilibrium with its surroundings. As for the second, if we recall that

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx,$$

we see that the limiting temperature $\frac{1}{2}a_0$ is simply the average value of the initial temperature. In other words, no heat enters or escapes, so the various parts of the interval simply come into thermal equilibrium with each other. Moreover, in both cases, the high-frequency terms (i.e., the terms with n large) damp out more quickly than the low-frequency terms: this expresses the fact that the diffusion of heat tends to quickly smooth out local variations in temperature. A simple illustration of these assertions can be found in Figure 2.6.

Now let us turn to the problem of the vibrating string:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad u(0,t) = u(l,t) = 0.$$

According to the discussion in §1.3, we should expand f and g in their Fourier sine series,

$$f(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l}, \qquad g(x) = \sum_1^\infty B_n \sin\frac{n\pi x}{l}, \tag{2.26}$$

and then take

$$u(x,t) = \sum_1^\infty \sin\frac{n\pi x}{l}\left(b_n \cos\frac{n\pi ct}{l} + \frac{lB_n}{n\pi c}\sin\frac{n\pi ct}{l}\right). \tag{2.27}$$

Here the analysis is more delicate than for the heat equation, because there are no exponentially decreasing factors in this series to help the convergence. The series (2.27) for u is likely to converge about as well as the sine series for f and g, but if we differentiate it twice with respect to x or t in order to verify the wave equation, we introduce a factor of n^2; and this may well be enough to destroy the convergence.

We can avoid this difficulty by making sufficiently strong smoothness assumptions on f and g. For instance, let us suppose that f and g are of class $C^{(3)}$ and $C^{(2)}$, respectively, that f''' and g'' are piecewise smooth, and that f, g, f'', and g'' vanish at the endpoints 0 and l. These conditions guarantee that the odd periodic extensions of f and g will have the same smoothness properties (even at the points $n\pi$), and hence, by Theorem 2.6, that the coefficients b_n and B_n will satisfy

$$|b_n| \le Cn^{-4}, \qquad |B_n| \le Cn^{-3}.$$

Now the nth term in the series (2.27) will be dominated by n^{-4}, and if we differentiate it twice in either x or t it is still dominated by n^{-2}. Since $\sum_1^\infty n^{-2}$

converges, the M-test guarantees the absolute and uniform convergence of the twice-derived series, and we are in business.

This is not entirely satisfactory, however. It is physically reasonable to assume that f and g are continuous and perhaps piecewise smooth, but one may — and indeed should — have the feeling that the extra differentiability assumptions are annoyances that reflect a failure of technique rather than a real difficulty in the original problem.

We can obtain more insight into this problem by recalling the trigonometric identities

$$\sin a \cos b = \tfrac{1}{2}\Big[\sin(a+b)+\sin(a-b)\Big], \qquad \sin a \sin b = \tfrac{1}{2}\Big[\cos(a-b)-\cos(a+b)\Big],$$

by means of which the series (2.27) can be rewritten

$$\begin{aligned} u(x,t) =& \frac{1}{2}\sum_1^\infty b_n \sin\frac{n\pi}{l}(x+ct) + \frac{1}{2}\sum_1^\infty b_n \sin\frac{n\pi}{l}(x-ct) \\ &+ \frac{1}{2c}\sum_1^\infty \frac{lB_n}{n\pi}\cos\frac{n\pi}{l}(x-ct) - \frac{1}{2c}\sum_1^\infty \frac{lB_n}{n\pi}\cos\frac{n\pi}{l}(x+ct). \end{aligned}$$

The first two sums on the right are just the Fourier sine series for f, evaluated at $x \pm ct$, and the last two are (up to constant factors) just the Fourier sine series for g, integrated once and then evaluated at $x \pm ct$. To restate this: let us suppose that f and g are piecewise smooth, so that the expansions (2.26) are valid on the interval $(0,l)$. We use the formulas (2.26) to extend f and g from this interval to the whole line; that is, we extend f and g to $\mathbf{R}$ by requiring them to be odd and $2l$-periodic. We then have

$$u(x,t) = \frac{1}{2}\Big[f(x+ct)+f(x-ct)\Big] + \frac{1}{2c}\Big[G(x+ct)-G(x-ct)\Big], \tag{2.28}$$

where G is any antiderivative of g.

From this closed formula it is perfectly plain that if f is twice differentiable and g is once differentiable, then u satisfies the wave equation, for

$$\frac{\partial^2}{\partial x^2}f(x\pm ct) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}f(x\pm ct) = f''(x\pm ct), \tag{2.29}$$

and likewise for G. Even here the differentiability assumptions seem a bit artificial; one would like, for example, to allow f to be a function with corners in order to model plucked strings. Indeed, in some sense the first equation in (2.29) should be correct, simply as a formal consequence of the chain rule, even if f'' is ill-defined. The idea that is crying to be set free here is the notion of a "weak solution" of a differential equation, which enables one to consider functions u defined by (2.28) as solutions of the wave equation even when the requisite derivatives of f and g do not exist. We shall say more about this in §9.5.

Another point should be raised here. One does not have to go through Fourier series to produce the formula (2.28) for the solution of the vibrating string problem; an elementary derivation is sketched in Exercise 6 of §1.1. It is then fair to ask what good the complicated-looking formula (2.27) is when the simple (2.28) is readily available. There are two good answers. First, the trick in Exercise 6, §1.1, that quickly produces the general solution of the 1-dimensional wave equation does not work for other equations (including the higher-dimensional wave equation), whereas the Fourier method and its generalizations often do. Second, although (2.28) tells you what you see if you look at a vibrating string, (2.27) *tells you what you hear when you listen to it.* The ear, unlike the eye, has a built-in Fourier analyzer that resolves sound waves into their components of different frequencies, which are perceived as musical tones.* Typically, the first term in the series (2.27) is the largest one, so one hears the note with frequency $2\pi c/l$ colored by the "overtones" at the higher frequencies $2\pi nc/l$ with $n > 1$.

The difference in the convergence properties of the series solutions (2.25) and (2.27) of the heat and wave equations reflects a difference in the physics: diffusion processes such as heat flow tend to smooth out any irregularities in the initial data, whereas wave motion propagates singularities. Thus, the solution (2.25) of the heat equation becomes smoother as t increases, and this is reflected in the exponential decay of the high-frequency terms. (See the discussion of smoothness versus rates of convergence at the end of §2.3.) However, any sharp corners in the initial configuration of a vibrating string will not disappear but merely move back and forth, as is clear from (2.28); hence there is no improvement in the rate of convergence of the solution (2.27). (Compare Figures 2.6 and 2.7, which show solutions of the heat and wave equations with the same initial values up to a constant factor and the same boundary conditions; the initial variations damp out in the first case, but not in the second.)

We shall see other applications of Fourier expansions of functions on an interval in Chapter 4. Fourier expansions are also the natural tool for analyzing periodic functions on the line. In practice, there are two principal sources of such functions. The first is the angular variable in polar or cylindrical coordinates or the longitudinal angular variable in spherical coordinates; in this context periodicity is an immediate consequence of the geometry of the situation. The other is physical phenomena that vary periodically (or approximately periodically) in time, such as certain types of electrical signals, the length of a day, daily or seasonal variations in temperature, and so forth.

As an example, let us analyze the variations in temperature beneath the ground due to the daily and seasonal fluctuations of temperature at the surface of the earth. We shall concern ourselves only with the temperature near a particular spot on the surface, over distances of (say) at most 100 meters. We therefore neglect the fact that at great depths the earth is hotter than at the surface, and we assume that (i) the earth is of uniform composition; (ii) the temperature at the surface is a function $f(t)$ of time only, not of position; (iii) $f(t)$ is periodic

* Of course, this is an oversimplification.

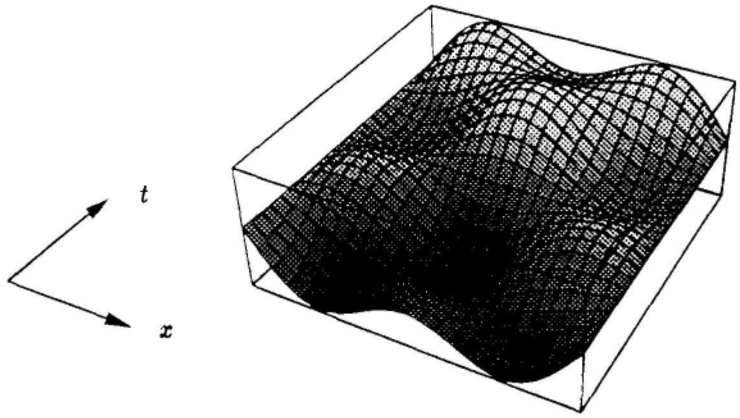


FIGURE 2.7. The solution (2.27) of the wave equation with $l = c = 1$, $b_1 = -0.2$, $b_2 = -0.1$, $b_n = 0$ for $n > 2$, and $B_n = 0$ for all n, on the region $0 \le x \le 1$, $0 \le t \le 1$.

of period 1 and so has a Fourier series

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{2\pi i n t}.$$

(We may take the unit of time to be 1 year, so that the dominant terms in the series will be $n = \pm 1$, corresponding to seasonal variations, and $n = \pm 365$, corresponding to daily variations. With a bit more accuracy, we could take the unit of time to be 4 years and the dominant terms to be $n = \pm 4$ and $n = \pm 1461$ ($= \pm 4 \times 365\frac{1}{4}$). Or, we could take an even longer period to account for long-term climatic changes.) The boundary value problem to be solved is therefore

$$u_t = k u_{xx} \quad \text{for } x > 0, \qquad u(0,t) = f(t).$$

Since f is periodic in t, we expect u to have the same property, so we look for solutions of the form

$$u(x,t) = \sum_{-\infty}^{\infty} C_n(x) e^{2\pi i n t}.$$

Taking on faith that this series can be differentiated termwise, we find that

$$u_t = \sum_{-\infty}^{\infty} (2\pi i n) C_n(x) e^{2\pi i n t}, \qquad u_{xx} = \sum_{-\infty}^{\infty} C_n''(x) e^{2\pi i n t}.$$

Hence, taking into account the initial condition, we have

$$C_n''(x) - 2\pi i n k^{-1} C_n(x) = 0, \qquad C_n(0) = c_n.$$

Since the square roots of $2in$ are $\pm(1+i)n^{1/2}$ if $n > 0$ and $\pm(1-i)|n|^{1/2}$ if $n < 0$, the general solution of this differential equation is

$$\begin{aligned} a\exp\left((1+i)\sqrt{\frac{\pi n}{k}}\,x\right) + b\exp\left(-(1+i)\sqrt{\frac{\pi n}{k}}\,x\right) &\quad \text{if} \quad n > 0, \\ a\exp\left((1-i)\sqrt{\frac{\pi|n|}{k}}\,x\right) + b\exp\left(-(1-i)\sqrt{\frac{\pi|n|}{k}}\,x\right) &\quad \text{if} \quad n < 0, \\ ax + b &\quad \text{if} \quad n = 0. \end{aligned}$$

In each case we must take $a = 0$ because of the physical requirement that the temperature should remain bounded as x increases. (In effect we are imposing a boundary condition at $x = \infty$ to supplement the one at $x = 0$.) The initial condition then implies that $b = c_n$. Hence, upon grouping together the nth and $(-n)$th terms, we obtain the solution

$$u(x,t) = c_0 + \sum_1^\infty \exp\left(-\sqrt{\frac{\pi n}{k}}\,x\right) \\ \times \left[c_n \exp\left(2\pi i n t - i\sqrt{\frac{\pi n}{k}}\,x\right) + c_{-n}\exp\left(-2\pi i n t + i\sqrt{\frac{\pi n}{k}}\,x\right)\right].$$

It is now easy to check that this function u really does solve the problem.

The main features to be noted here are the following. First, all of the non-constant terms in u (the ones with $n \neq 0$) die out exponentially fast as x increases, and the high-frequency ones die out faster than the low-frequency ones. (In actual fact, the daily variations in temperature become negligible at a depth of a few centimeters, and the seasonal ones become negligible at a depth of a few meters, where the temperature remains essentially constant at the annual mean c_0.) Second, the temperature variations at depth x are out of phase with those at the surface by an amount proportional to x and $\sqrt{|n|}$, because the heat takes time to penetrate. For example, if the $n = 1$ term, representing the main seasonal variations, is the dominant one, at depth $x = \sqrt{\pi k}$ the temperature is warmer in winter and cooler in summer.

In considering the usefulness of Fourier series or any other sort of infinite series, one should not lose sight of the fact that the partial sums of the series provide approximations to the full sum, and that such approximations may be just what one needs to obtain a computationally manageable solution to a problem. The questions about smoothness and rates of convergence that we have discussed in some detail have a computational as well as a theoretical significance: rapidly converging series such as (2.25) yield accurate answers much more readily than slowly converging ones such as (2.27). An interesting discussion of rates of convergence of infinite series, and the implications for numerical calculations, can be found in Boas [7].

On the other hand, in many situations one knows the initial data only to a finite degree of accuracy. For example, one may be studying a physical quantity

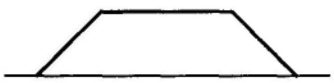
$f(t)$ that varies periodically with the time t, and one may know the values of f approximately from physical measurements. In this context the point of Fourier analysis is that it is usually appropriate to take a trigonometric polynomial of fairly low degree, whose coefficients are determined so as to fit the data well, as a mathematical model for f.

EXERCISES

1. A rod 100 cm long is insulated along its length and at both ends. Suppose that its initial temperature is $u(x,0) = x$ (x in cm, u in °C, t in sec, $0 \le x \le 100$), and that its diffusivity coefficient k is 1.1 cm^2/sec (about right if the bar is made of copper).
   a. Find the temperature $u(x,t)$ for $t > 0$. (It is something of the form $50 + \sum_1^\infty a_n(t)\cos(n\pi x/100)$, and $a_n(t) = 0$ when n is even.)
   b. Show that the first three terms of the series (i.e., $50 + a_1(t)\cos(\pi x/100) + a_3(t)\cos(3\pi x/100)$) give the temperature accurately to within 1 unit when $t = 60$. Using this fact, find $u(0,60)$, $u(10,60)$, and $u(40,60)$.

   $$\text{Hint: } \sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}, \quad \text{so} \quad \sum_3^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - 1 - \frac{1}{9} \approx .123.$$

   c. Find a number $T > 0$ such that $u(x,t)$ is within 1 unit of its equilibrium value 50 for all x when $t > T$.
2. Redo Exercises 1a and 1c with $k = .01$ (a reasonable figure if the bar is made of ceramic). Now how many terms of the series are needed to get an accuracy of 1 unit when $t = 60$?
3. Consider again the copper rod of Exercise 1 ($k = 1.1$). Suppose that the rod is initially at temperature 100°C and that the ends are subsequently put into a bath of ice water (at 0°C).
   a. Assuming no heat loss along the length of the rod, find the temperature $u(x,t)$ at subsequent times.
   b. Use your answer to find $u(50,t)$ numerically when $t = 30, 60, 300, 3600$.
   c. Prove that your answers in (b) are correct to within 1 unit. (Hint: The series for $u(50,t)$ is alternating.)
4. Consider a vibrating string occupying the interval $0 \le x \le l$. Suppose the string is plucked in the middle in such a way that its initial displacement $u(x,0)$ is $2mx/l$ for $0 \le x \le \frac{1}{2}l$ and $2m(l-x)/l$ for $\frac{1}{2}l \le x \le l$ (so the maximum displacement, at $x = \frac{1}{2}l$, is m), and its initial velocity $u_t(x,0)$ is zero.
   a. Find the displacement $u(x,t)$ as a Fourier series.
   b. Describe $u(x,t)$ in the closed form (2.28) and show that at times $t > 0$, $u(x,t)$ (as a function of x) typically looks like the following figure:



or

5. Consider a vibrating string as in Exercise 4. Suppose the string is plucked at $x = a$ instead of $x = \frac{1}{2}l$, so the initial displacement is mx/a for $0 \leq x \leq a$ and $m(l-x)/(l-a)$ for $a \leq x \leq l$, and the initial velocity is zero.
   a. Find the displacement $u(x,t)$ as a Fourier series. (Entry 11 of Table 1, §2.1, will be helpful.)
   b. Convince yourself that the terms with large n contribute more to $u(x,t)$ when a becomes closer to l. (Musically: plucking near the end gives a tone with more higher harmonics.)
6. Suppose the string in Exercise 4 is initially struck in the middle so that its initial displacement is zero but its initial velocity $u_t(x,0)$ is 1 for $|x - \frac{1}{2}l| < \delta$ and 0 elsewhere. Find $u(x,t)$ for $t > 0$.
7. Suppose that the temperature at time t at a point on the surface of the earth is given by
$$u(0,t) = 10 - 7\cos 2\pi t - 5\cos 2\pi(365)t.$$
(Here u is measured in $°C$ and t is measured in years; the coefficients are roughly correct for Seattle, Washington.) Suppose that the diffusivity coefficient of the earth is $k = .003\ \text{cm}^2/\text{sec} \approx 9.46\ \text{m}^2/\text{yr}$.
   a. Find $u(x,t)$ for $x > 0$.
   b. At what depth x do the daily variations in temperature become less than 1 unit? What about the annual variations?

2.6 Further remarks on Fourier series

There is much more to be said about Fourier series than is contained in this chapter. Some good references for further information on both the theoretical aspects of the subject and its applications are the books of Dym-McKean [19], Körner [34], and Walker [53]. Also recommended is the article of Coppel [15] on the history of Fourier analysis and its influence on other branches of mathematics, and the articles by Zygmund, Hunt, and Ash in [2]. Finally, the serious student of Fourier analysis should become acquainted with the treatise of Zygmund [58], which gives an encyclopedic account of the subject.

We conclude this chapter with a brief discussion of a few other interesting aspects of Fourier series.

The transform point of view

Given a 2π-periodic function, its sequence $\{c_n\}$ of Fourier coefficients can be regarded as a function $\widehat{f}$ whose domain is the integers:

$$\widehat{f}(n) = c_n = \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta.$$

The mapping $f \to \widehat{f}$ is thus a *transform* that converts periodic functions on the line to functions on the integers. The inverse transform is the operation which assigns to a function $\phi(n)$ on the integers (that decays suitably as $n \to \infty$) the function $\sum_{-\infty}^{\infty} \phi(n)e^{in\theta}$. In principle all the information in f is also contained in its transform $\widehat{f}$, and vice versa, but the information may be encoded in a more convenient form on one side or the other. For example, Theorem 2.2 shows that the transform converts differentiation into a simple algebraic operation: $\widehat{f'}(n) = in\widehat{f}(n)$. We shall return to this point of view in Chapter 7.

Comparison with Taylor series

Perhaps the most well known and widely used type of infinite series expansion for functions is the Taylor series, and it is of interest to compare the features of Taylor series and Fourier series.

In order for a function $f(x)$ to have a Taylor expansion about a point x_0,

$$f(x) = \sum_0^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \qquad |x - x_0| < r,$$

f must have derivatives of all orders at x_0. If it does, the coefficients of the Taylor series are determined by these derivatives, and hence by the values of f in an arbitrarily small neighborhood of x_0. The rate at which these coefficients grow or decay as $n \to \infty$ is related to the radius of convergence of the series and hence to the distance from x_0 to the nearest singularity of f (in the complex plane). In general the partial sums of the Taylor series provide excellent approximations to f near x_0 but are often of little use when $|x - x_0|$ is large.

In contrast, a function f need have only minimal smoothness properties in order to have a convergent Fourier expansion

$$f(x) = \sum_{-\infty}^{\infty} \left((2l)^{-1} \textstyle\int_a^{a+2l} f(y)e^{-in\pi y/l}dy\right) e^{in\pi x/l}, \qquad x \in (a,\, a+2l).$$

The coefficients of this series depend on the values of f over the entire interval $(a,\, a+2l)$. The rate at which they decay as $n \to \infty$ is related to the differentiability properties of f, or rather of its periodic extension. The partial sums of the Fourier series will converge to f only rather slowly if f is not very smooth, but they tend to provide good approximations over the whole interval $(a, a + 2l)$.

Thus Taylor series and Fourier series are of quite different natures: the first one is intimately connected with the local properties of f near x_0, whereas the second is related to global properties of f. There is a situation, however, in which the two can be seen as aspects of the same thing. Namely, suppose f is an analytic function of the complex variable z in some disc $|z - z_0| < R$. If we write $z - z_0$ in polar coordinates as $re^{i\theta}$, the Taylor series for f about z_0 turns into a Fourier series in θ for each fixed $r < R$:

$$\sum_0^\infty a_n(z - z_0)^n = \sum_0^\infty (a_n r^n)e^{in\theta}.$$

The formula (2.5) for the Fourier coefficients, in this case, is nothing but the Cauchy integral formula for the derivatives of f at z_0. This connection between Fourier analysis and complex function theory has many interesting consequences, which are discussed in more advanced books such as Dym-McKean [19] and Zygmund [58].

Convergence of Fourier series

The study of the convergence of Fourier series has a long and complex history. The convergence theorems we have presented in §§2.2–3 are sufficient for many purposes, but they do not give the whole picture. Here we briefly indicate a few other highlights of the story. In the first place, the hypotheses of our Theorem 2.1 can be weakened. The same conclusion is obtained if we assume only that f is of "bounded variation" on the interval $[-\pi, \pi]$, which means that it can be written as the difference of two nondecreasing functions on that interval. (It is not hard to show that piecewise smooth functions have this property.) On the other hand, it has been known since 1876 that there are continuous periodic functions whose Fourier series diverge at some points, and for almost a century it was an open question whether the Fourier series of a continuous function could be guaranteed to converge at *any* point. An affirmative answer was obtained only in 1966, with a deep theorem of L. Carleson to the effect that the Fourier series of any square-integrable function f must converge to f at "almost every" point, in a sense that we shall describe in §3.3. See the article by Hunt in [2].

One fundamental fact that has emerged over the years is that, in many situations, simple pointwise convergence of a series is not the appropriate thing to look at; and there are many other notions of convergence that may be used. For example, there is uniform convergence, which is stronger than pointwise convergence; there is also "pth power mean" convergence, according to which the series $\sum_1^\infty f_n$ converges to f on the interval $[a, b]$ if

$$\lim_{N\to\infty} \int_a^b \left| \sum_1^N f_n(x) - f(x) \right|^p dx = 0.$$

We shall say much more about the case $p = 2$ in the next chapter. There are also ways of summing divergent series that can be used to advantage; we shall now briefly discuss the simplest of these.

It is easy to see that if a sequence $\{a_n\}$ converges to a, then the average $k^{-1}\sum_1^k a_n$ of its first k terms also converges to a as $k \to \infty$, but these averages may converge when the original sequence does not. For example, the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

is divergent; but the average of its first k terms is $(k+1)/2k$ or $1/2$ according as k is odd or even, and this tends to $1/2$ as $k \to \infty$. Now, given an infinite series $\sum_0^\infty b_n$ with partial sums $s_N = \sum_0^N b_n$, the average of its first $k+1$ partial sums,

$$\frac{1}{k+1}(s_0 + s_1 + \cdots + s_k),$$

is called its kth **Cesàro mean**, and the series is said to be **Cesàro summable** to the number s if its Cesàro means (rather than just its partial sums) converge to s. We then have the following theorem, due to L. Fejér.

Theorem 2.8. *If f is 2π-periodic and piecewise continuous on* **R**, *then the Fourier series of f is Cesàro summable to $\frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$ at every θ. Moreover, if f is everywhere continuous, the Cesàro means of the series converge to f uniformly.*

The proof of this theorem is similar in spirit to that of Theorem 2.1; it can be found, for example, in §2 of Körner [34] or §2.7 of Walker [53]. The significance of the theorem is twofold. First, it gives a way of recovering a piecewise continuous function f from its Fourier coefficients when the Fourier series fails to converge. Second, even when the Fourier series of f does converge, its Cesáro means tend to give better approximations to f than its partial sums: for example, they converge uniformly to f whenever f is continuous, whereas the partial sums converge uniformly only under stronger smoothness conditions (cf. Theorem 2.5).

The Gibbs phenomenon

Suppose f is a periodic function. If f has a discontinuity at x_0, the Fourier series of f cannot converge uniformly on any interval containing x_0, because the uniform limit of continuous functions is continuous. In fact, for the Fourier series of a piecewise smooth function f, the lack of uniformity manifests itself in a particularly dramatic way known as the **Gibbs phenomenon**: as one adds on more and more terms, the partial sums overshoot and undershoot f near the discontinuity and thus develop "spikes" that tend to zero in width but *not* in height. One can see this in Figure 2.8, which shows the fortieth partial sum of the Fourier series of the sawtooth wave function

$$f(\theta) = \pi - \theta \text{ for } 0 < \theta < 2\pi, \qquad f(\theta + 2n\pi) = f(\theta).$$

A precise statement and proof of the Gibbs phenomenon for this function is outlined in Exercise 1. It can be shown that the same behavior occurs at any discontinuity of any piecewise smooth function. See Körner [34] and Hewitt-Hewitt [28] for interesting discussions of the Gibbs phenomenon.

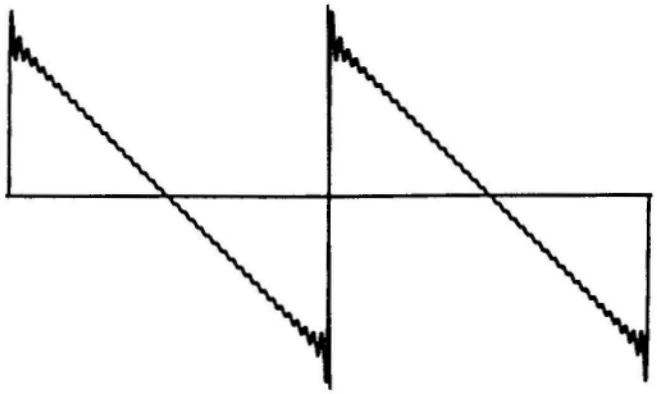


FIGURE 2.8. Graph of $2\sum_1^{40} n^{-1} \sin n\theta$, $-2\pi < \theta < 2\pi$ (an illustration of the Gibbs phenomenon).

EXERCISE

1. Recall from Table 1, §2.1, that $f(\theta) = 2\sum_1^\infty n^{-1} \sin n\theta$ is the 2π-periodic function that equals $\pi - \theta$ for $0 < \theta < 2\pi$. Let

$$g_N(\theta) = 2\sum_1^N \frac{\sin n\theta}{n} - (\pi - \theta),$$

so that $g(\theta)$ is the difference between $f(\theta)$ and its Nth partial sum for $0 < \theta < 2\pi$.

a. Show that $g_N'(\theta) = 2\pi D_N(\theta)$ where D_N is the Dirichlet kernel (2.10).

b. Using (2.12), show that the first critical point of $g_N(\theta)$ to the right of zero occurs at $\theta_N = \pi/(N + \frac{1}{2})$, and that

$$g_N(\theta_N) = \int_0^{\theta_N} \frac{\sin(N + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta} d\theta - \pi.$$

c. Show that

$$\lim_{N\to\infty} g_N(\theta_N) = 2\int_0^\pi \frac{\sin \phi}{\phi} d\phi - \pi.$$

(Hint: Let $\phi = (N + \frac{1}{2})\theta$.) This limit is approximately equal to .562. Thus the difference between $f(\theta)$ and the Nth partial sum of its Fourier series develops a spike of height .562 (but of increasingly narrow width) just to the right of $\theta = 0$ as $N \to \infty$. (There is another such spike on the left.)

CHAPTER 3
ORTHOGONAL SETS OF FUNCTIONS

Fourier series are only one of a large class of interesting and useful infinite series expansions for functions that are based on so-called *orthogonal systems* or *orthogonal sets* of functions. This chapter is devoted to explaining the general conceptual framework for understanding such systems, and to showing how they arise from certain kinds of differential equations. Underlying these ideas is a profound analogy between the algebra of Fourier series and the algebra of n-dimensional vectors, which we now investigate.

3.1 Vectors and inner products

We recall some ideas from elementary 3-dimensional vector algebra and recast them in a more general form. We identify 3-dimensional vectors with ordered triples of real numbers; that is, we write

$$\mathbf{a} = (a_1, a_2, a_3) \qquad \text{rather than} \qquad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

The dot product or inner product of two vectors is then defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

and the norm or length of a vector is defined by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

We propose to generalize these ideas in two ways: by working in an arbitrary number k of dimensions, and by using complex numbers rather than real ones. This generalization is not just a mathematical fantasy. Although k-dimensional vectors do not have an immediate geometrical interpretation in physical space, they are still useful for dealing with problems involving k independent variables. For our purposes, the main motivation for the use of complex numbers is their connection with the exponentials $e^{i\theta}$; but it should be noted that the use of complex vectors is essential in quantum physics. However, in visualizing the ideas we shall be discussing, the reader should just think of real 3-dimensional vectors.

A **(complex)** k**-dimensional vector** is an ordered k-tuple of complex numbers:

$$\mathbf{a} = (a_1, a_2, \ldots, a_k).$$

The vector $\mathbf{a}$ is called **real** if its components a_j are all real numbers. Addition and scalar multiplication are defined just as in the 3-dimensional case, but now the scalars are allowed to be complex:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, \ldots, a_k + b_k),$$
$$c\mathbf{a} = (ca_1, \ldots, ca_k) \qquad (c \in \mathbf{C}).$$

We denote the zero vector $(0, 0, \ldots, 0)$ by $\mathbf{0}$, and we denote the space of all complex k-dimensional vectors by $\mathbf{C}^k$.

The **inner product** of two vectors is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1\overline{b}_1 + a_2\overline{b}_2 + \cdots + a_k\overline{b}_k, \tag{3.1}$$

and the **norm** of a vector is defined by

$$\|\mathbf{a}\| = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2} = \left(a_1\overline{a}_1 + \cdots + a_k\overline{a}_k\right)^{1/2} = \left(|a_1|^2 + \cdots + |a_k|^2\right)^{1/2}. \tag{3.2}$$

The reason for the complex conjugates in the definition of the inner product is to make the norm (3.2) positive, for we wish to interpret $\|\mathbf{a}\|$ as the magnitude or length of the vector $\mathbf{a}$. (Recall that the absolute value of a complex number $z = x + iy$ is $(x^2+y^2)^{1/2}$, and this is $(z\overline{z})^{1/2}$ rather than $(z^2)^{1/2}$.) Notice, however, that for real vectors, (3.1) and (3.2) become

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1b_1 + \cdots + a_kb_k, \qquad \|\mathbf{a}\| = (a_1^2 + \cdots + a_k^2)^{1/2},$$

the obvious generalization of the familiar 3-dimensional case.

A word about the notation: The inner product $\langle \mathbf{a}, \mathbf{b} \rangle$ is often denoted by $\mathbf{a} \cdot \mathbf{b}$ or $(\mathbf{a}, \mathbf{b})$. Also, in the physics literature it is customary to switch the roles of $\mathbf{a}$ and $\mathbf{b}$, that is, to put the complex conjugates on the first variable rather than the second. This discrepancy is regrettable, but by now it is firmly entrenched in common usage.

The inner product (3.1) is clearly linear as a function of its first variable but *antilinear* or *conjugate linear* as a function of its second variable; that is, for any vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and any complex numbers z, w,

$$\begin{aligned} \langle z\mathbf{a} + w\mathbf{b}, \mathbf{c} \rangle &= z\langle \mathbf{a}, \mathbf{c} \rangle + w\langle \mathbf{b}, \mathbf{c} \rangle, \\ \langle \mathbf{a}, z\mathbf{b} + w\mathbf{c} \rangle &= \overline{z}\langle \mathbf{a}, \mathbf{b} \rangle + \overline{w}\langle \mathbf{a}, \mathbf{c} \rangle \end{aligned} \tag{3.3}$$

Also, the inner product is *Hermitian symmetric*, which means that

$$\langle \mathbf{b}, \mathbf{a} \rangle = \overline{\langle \mathbf{a}, \mathbf{b} \rangle}, \tag{3.4}$$

and the norm satisfies the conditions

$$\|c\mathbf{a}\| = |c|\,\|\mathbf{a}\| \qquad (c \in \mathbf{C}), \tag{3.5}$$
$$\|\mathbf{a}\| > 0 \quad \text{for all } \mathbf{a} \neq \mathbf{0}. \tag{3.6}$$

Using these facts, we now derive some fundamental properties of inner products and norms.

Lemma 3.1. *For any* **a** *and* **b** *in* $\mathbf{C}^k$,

$$\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2\operatorname{Re}\langle \mathbf{a},\mathbf{b}\rangle + \|\mathbf{b}\|^2.$$

Proof: By (3.3), (3.4), and the definition of the norm,

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= \langle \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}\rangle = \langle \mathbf{a},\mathbf{a}\rangle + \langle \mathbf{a},\mathbf{b}\rangle + \langle \mathbf{b},\mathbf{a}\rangle + \langle \mathbf{b},\mathbf{b}\rangle \\ &= \langle \mathbf{a},\mathbf{a}\rangle + \langle \mathbf{a},\mathbf{b}\rangle + \overline{\langle \mathbf{a},\mathbf{b}\rangle} + \langle \mathbf{b},\mathbf{b}\rangle = \|\mathbf{a}\|^2 + 2\operatorname{Re}\langle \mathbf{a},\mathbf{b}\rangle + \|\mathbf{b}\|^2.\end{aligned}$$ ∎

The Cauchy-Schwarz Inequality. *For any* **a** *and* **b** *in* $\mathbf{C}^k$,

$$\left|\langle \mathbf{a},\mathbf{b}\rangle\right| \le \|\mathbf{a}\|\,\|\mathbf{b}\|. \tag{3.7}$$

Proof: We may assume that $\mathbf{b} \neq \mathbf{0}$, since otherwise both sides of (3.7) are 0. Also, neither $\left|\langle \mathbf{a},\mathbf{b}\rangle\right|$ nor $\|\mathbf{a}\|\,\|\mathbf{b}\|$ is affected if we multiply **a** by a scalar of absolute value one, so we may replace **a** by $c\mathbf{a}$, with $|c| = 1$, so as to make $\langle \mathbf{a},\mathbf{b}\rangle$ real. (That is, if $\langle \mathbf{a},\mathbf{b}\rangle = re^{i\theta}$, we take $c = e^{-i\theta}$.) Assuming then that $\langle \mathbf{a},\mathbf{b}\rangle$ is real, by Lemma 3.1 we see that for any real number t,

$$0 \le \|\mathbf{a}+t\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2t\langle \mathbf{a},\mathbf{b}\rangle + t^2\|\mathbf{b}\|^2.$$

This last expression is a quadratic function of t, since $\|\mathbf{b}\| \neq 0$, and (by elementary calculus) it achieves its minimum value at $t = -\langle \mathbf{a},\mathbf{b}\rangle/\|\mathbf{b}\|^2$. If we substitute this value for t, we obtain

$$0 \le \|\mathbf{a}\|^2 - 2\frac{\langle \mathbf{a},\mathbf{b}\rangle^2}{\|\mathbf{b}\|^2} + \frac{\langle \mathbf{a},\mathbf{b}\rangle^2}{\|\mathbf{b}\|^4}\|\mathbf{b}\|^2 = \|\mathbf{a}\|^2 - \frac{\langle \mathbf{a},\mathbf{b}\rangle^2}{\|\mathbf{b}\|^2},$$

or

$$0 \le \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \langle \mathbf{a},\mathbf{b}\rangle^2,$$

which, since $\langle \mathbf{a},\mathbf{b}\rangle$ is assumed real, is equivalent to (3.7). ∎

The Triangle Inequality. *For any* **a** *and* **b** *in* $\mathbf{C}^k$,

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|. \tag{3.8}$$

Proof: By Lemma 3.1, the Cauchy-Schwarz inequality, and the fact that $\operatorname{Re} z \le |z|$, we have

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + 2\operatorname{Re}\langle \mathbf{a},\mathbf{b}\rangle + \|\mathbf{b}\|^2 \\ &\le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\,\|\mathbf{b}\| + \|\mathbf{b}\|^2 \\ &= \left(\|\mathbf{a}\| + \|\mathbf{b}\|\right)^2.\end{aligned}$$ ∎

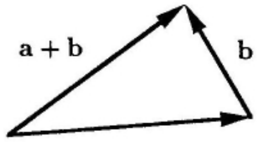


FIGURE 3.1. The sum of two vectors.

Geometrically, the triangle inequality just says that one side of a triangle can be no longer than the sum of the other two sides; see Figure 3.1. This picture is perfectly accurate, for the vectors **a**, **b**, and **a** + **b** always lie in the same plane no matter how many dimensions they live in.

We recall that two real 3-dimensional vectors are orthogonal or perpendicular to each other precisely when their inner product is zero. We shall take this as a definition in the general case: two complex k-dimensional vectors **a** and **b** are **orthogonal** if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$. The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are called **mutually orthogonal** if $\langle \mathbf{a}_i, \mathbf{a}_j \rangle = 0$ for all $i \neq j$. With this terminology, we have a generalization of the classic theorem about the lengths of the sides of a right triangle:

The Pythagorean Theorem. *If* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *are mutually orthogonal, then*

$$\|\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_n\|^2 = \|\mathbf{a}_1\|^2 + \|\mathbf{a}_2\|^2 + \cdots + \|\mathbf{a}_n\|^2. \tag{3.9}$$

Proof: We have

$$\|\mathbf{a}_1 + \cdots + \mathbf{a}_n\|^2 = \langle \mathbf{a}_1 + \cdots + \mathbf{a}_n,\ \mathbf{a}_1 + \cdots + \mathbf{a}_n \rangle.$$

If we multiply out the right side by (3.3), all the cross terms vanish because of the orthogonality condition, and we are left with

$$\langle \mathbf{a}_1, \mathbf{a}_1 \rangle + \cdots + \langle \mathbf{a}_n, \mathbf{a}_n \rangle = \|\mathbf{a}_1\|^2 + \cdots + \|\mathbf{a}_n\|^2. \qquad \blacksquare$$

Important Remark. The proofs of the Cauchy-Schwarz and triangle inequalities and the Pythagorean theorem depend only on the properties (3.3) and (3.4) of the inner product and the definition $\|\mathbf{a}\| = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2}$, not on the specific formula (3.1). They therefore remain valid for any other "inner product" that satisfies (3.3) and (3.4) and the "norm" associated to it.

Some more terminology: We say that a vector **u** is **normalized**, or is a **unit vector**, if $\|\mathbf{u}\| = 1$. Any nonzero vector **a** can be normalized by multiplying it by the reciprocal of its norm: If $\mathbf{u} = \|\mathbf{a}\|^{-1}\mathbf{a}$, then $\|\mathbf{u}\| = \|\mathbf{a}\|^{-1}\|\mathbf{a}\| = 1$. We shall call a collection $\{\mathbf{a}_1, \mathbf{a}_2, \ldots\}$ of vectors an **orthogonal set** if its elements are mutually orthogonal and nonzero, and an **orthonormal set** if its elements are mutually orthogonal and normalized. (See Figure 3.2.) Of course, any orthogonal set can be made into an orthonormal set by normalizing each of its elements. Thus, a set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots\}$ is orthonormal if and only if

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \delta_{ij}, \tag{3.10}$$

where δ_{ij} is the **Kronecker δ-symbol**:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.11}$$

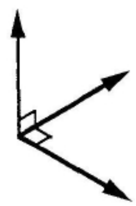


FIGURE 3.2. An orthonormal set of vectors.

The vectors in any orthogonal set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ are linearly independent; that is, the equation

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

can hold only when all the scalars c_j are zero. To see this, take the inner product of both sides with $\mathbf{a}_j$ $(1 \leq j \leq n)$; because of the orthogonality and the fact that $\mathbf{a}_j \neq \mathbf{0}$, the result is

$$c_j\langle \mathbf{a}_j, \mathbf{a}_j \rangle = c_j\|\mathbf{a}_j\|^2 = 0, \quad \text{hence} \quad c_j = 0.$$

It follows that *the number of vectors in any orthogonal set in $\mathbf{C}^k$ is at most k*, since $\mathbf{C}^k$ is k-dimensional.

An example of an orthonormal set of k vectors is given by the standard basis vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$, where

$$\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0) \qquad (1 \text{ in the } j\text{th position, } 0 \text{ elsewhere}).$$

For any $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbf{C}^k$, we clearly have

$$\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_k\mathbf{e}_k,$$

so $\mathbf{a}$ is expressed in a simple way as a linear combination of the $\mathbf{e}_j$'s. But sometimes it is more convenient to use other orthonormal sets that are adapted to a particular problem, and here too there is a simple way of expressing arbitrary vectors as linear combinations of the orthonormal vectors.

Indeed, suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal set in $\mathbf{C}^k$. If a vector $\mathbf{a} \in \mathbf{C}^k$ is expressed as a linear combination of the $\mathbf{u}_j$'s,

$$\mathbf{a} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k,$$

by taking the inner product of both sides with $\mathbf{u}_j$ and using (3.10) we find that the coefficients c_j are given by

$$c_j = \langle \mathbf{a}, \mathbf{u}_j \rangle \qquad (1 \le j \le k). \tag{3.12}$$

Conversely, if $\mathbf{a}$ is any vector in $\mathbf{C}^n$, we may define the constants $c_1, \ldots, c_k$ by (3.12) and form the linear combination

$$\widetilde{\mathbf{a}} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k.$$

Then the difference $\mathbf{b} = \mathbf{a} - \widetilde{\mathbf{a}}$ is orthogonal to all the u_j's:

$$\langle \mathbf{b}, \mathbf{u}_j \rangle = \langle \mathbf{a}, \mathbf{u}_j \rangle - \langle \widetilde{\mathbf{a}}, \mathbf{u}_j \rangle = c_j - c_j = 0.$$

But this means that $\mathbf{b} = \mathbf{0}$, for otherwise $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{b}\}$ would be an orthogonal set with $k + 1$ elements, which is impossible. In other words, $\widetilde{\mathbf{a}} = \mathbf{a}$, and we have the following result.

Theorem 3.1. *Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal set of k vectors in $\mathbf{C}^k$. For any $\mathbf{a} \in \mathbf{C}^k$ we have*

$$\mathbf{a} = \langle \mathbf{a}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \cdots + \langle \mathbf{a}, \mathbf{u}_k \rangle \mathbf{u}_k.$$

Moreover,

$$\|\mathbf{a}\|^2 = |\langle \mathbf{a}, \mathbf{u}_1 \rangle|^2 + \cdots + |\langle \mathbf{a}, \mathbf{u}_k \rangle|^2.$$

Proof: The first assertion has just been proved, and the second one follows from it by the Pythagorean theorem. ∎

EXERCISES

1. Show that $\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right)$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{C}^k$.
2. Suppose $\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ is an orthogonal set in $\mathbf{C}^k$, not necessarily normalized. Use Theorem 3.1 to show that for any $\mathbf{a} \in \mathbf{C}^k$,
$$\mathbf{a} = \frac{\langle \mathbf{a}, \mathbf{y}_1 \rangle \mathbf{y}_1}{\|\mathbf{y}_1\|^2} + \cdots + \frac{\langle \mathbf{a}, \mathbf{y}_k \rangle \mathbf{y}_k}{\|\mathbf{y}_k\|^2}.$$
3. Let $\mathbf{y}_1 = (2, 3i, 5)$ and $\mathbf{y}_2 = (3i, 2, 0)$.
   a. Show that $\langle \mathbf{y}_1, \mathbf{y}_2 \rangle = 0$ and find a nonzero $\mathbf{y}_3$ that is orthogonal to both $\mathbf{y}_1$ and $\mathbf{y}_2$.
   b. What are the norms of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$?
   c. Use Theorem 3.1 or Exercise 2 to express the vectors $(1, 2, 3i)$ and $(0, 1, 0)$ as linear combinations of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$.

4. Let $\mathbf{u}_1 = \frac{1}{3}(1, 2i, -2i, 0)$, $\mathbf{u}_2 = \frac{1}{5}(2-4i, -2, i, 0)$, $\mathbf{u}_3 = \frac{1}{15}(4+2i, 5+8i, 4+10i, 0)$, and $\mathbf{u}_4 = (0, 0, 0, i)$.
   a. Show that $\{\mathbf{u}_1, \dots, \mathbf{u}_4\}$ is an orthonormal set in $\mathbf{C}^4$.
   b. Express the vectors $(1, 0, 0, 0)$ and $(2, 10-i, 10-9i, -3)$ as linear combinations of $\mathbf{u}_1, \dots, \mathbf{u}_4$ by using Theorem 3.1.
5. Suppose $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is an orthonormal set in $\mathbf{C}^k$ with $m < k$. Show that for any $\mathbf{a} \in \mathbf{C}^k$ there is a unique set of constants $\{c_1, \dots, c_m\}$ such that $\mathbf{a} - \sum_1^m c_j \mathbf{u}_j$ is orthogonal to all the $\mathbf{u}_j$'s, and determine these constants explicitly. (Hint: Consider the proof of Theorem 3.1.)

The following problems deal with $k \times k$ complex matrices $T = (T_{ij})$. We recall that if $T = (T_{ij})$ and $S = (S_{ij})$ are $k \times k$ matrices, TS is the matrix whose (ij)th component is $\sum_l T_{il} S_{lj}$, and if $\mathbf{a} \in \mathbf{C}^k$, $T\mathbf{a}$ is the vector whose ith component is $\sum_j T_{ij} a_j$. The (Hermitian) **adjoint** of the matrix T is the matrix T^* obtained by interchanging rows and columns and taking complex conjugates, that is, $(T^*)_{ij} = \overline{T_{ji}}$.

6. Show that $\langle T\mathbf{a}, \mathbf{b}\rangle = \langle \mathbf{a}, T^*\mathbf{b}\rangle$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{C}^k$.
7. Show that if $T = T^*$, the "product" defined by $\langle \mathbf{a}, \mathbf{b}\rangle_T = \langle T\mathbf{a}, \mathbf{b}\rangle$ satisfies properties (3.3) and (3.4).
8. Let $\mathbf{t}_j = (T_{1j}, \dots, T_{kj})$ be the vector that makes up the jth row of T. Show that the following properties of the matrix T are equivalent. (Hint: Show that the (ij)th component of T^*T is $\langle \mathbf{t}_j, \mathbf{t}_i\rangle$.)
   (i) $\{\mathbf{t}_1, \dots, \mathbf{t}_k\}$ is an orthonormal basis for $\mathbf{C}^k$.
   (ii) T^*T is the identity matrix, i.e., $(T^*T)_{ij} = \delta_{ij}$.
   (iii) $\|T\mathbf{a}\| = \|\mathbf{a}\|$ for all $\mathbf{a} \in \mathbf{C}^k$.
9. Show that $|\langle \mathbf{a}, \mathbf{b}\rangle| = \|\mathbf{a}\|\,\|\mathbf{b}\|$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are complex scalar multiples of one another, and that $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are positive scalar multiples of one another. (Examine the proofs of the Cauchy-Schwarz and triangle inequalities to see when equality holds.)

3.2 Functions and inner products

A vector $\mathbf{a} = (a_1, \dots, a_k)$ in $\mathbf{C}^k$ can be regarded as a function on the set $\{1, \dots, k\}$ that assigns to the integer j the jth component $\mathbf{a}(j) = a_j$, and with this notation we can write the inner product and norm as follows:

$$\langle \mathbf{a}, \mathbf{b}\rangle = \sum_1^k \mathbf{a}(j)\overline{\mathbf{b}(j)}, \qquad \|\mathbf{a}\| = \left(\sum_1^k |\mathbf{a}(j)|^2\right)^{1/2}. \tag{3.13}$$

We now make a leap of imagination: Consider the space $PC(a, b)$ of piecewise continuous functions on the interval $[a, b]$, and think of functions $f \in PC(a, b)$ as infinite-dimensional vectors whose "components" are the values $f(x)$ as x ranges over the interval $[a, b]$. The operations of vector addition and scalar multiplication are just the usual addition of functions and multiplication of functions by constants. To define the inner product and the norm, we simply replace the sums in (3.13) by their continuous versions, i.e., integrals:

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx, \qquad \|f\| = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}. \tag{3.14}$$

This inner product on functions evidently satisfies the linearity and symmetry properties (3.3) and (3.4), and it is related to the norm by the equation $\|f\| = \langle f, f\rangle^{1/2}$. *Hence the Cauchy-Schwarz inequality, the triangle inequality, and the Pythagorean theorem remain valid in this context, with the same proofs.* Explicitly, in terms of integrals, they say the following:

$$\left|\int_a^b f(x)\overline{g(x)}\,dx\right| \le \sqrt{\int_a^b |f(x)|^2\,dx}\sqrt{\int_a^b |g(x)|^2\,dx}, \tag{3.15}$$

$$\sqrt{\int_a^b |f(x)+g(x)|^2\,dx} \le \sqrt{\int_a^b |f(x)|^2\,dx} + \sqrt{\int_a^b |g(x)|^2\,dx}, \tag{3.16}$$

and

$$\int_a^b \left|\sum_1^n f_j(x)\right|^2 dx = \sum_1^n \int_a^b |f_j(x)|^2\,dx \quad \text{when} \quad \int_a^b f_i(x)\overline{f_j(x)}\,dx = 0 \quad \text{for } i \neq j. \tag{3.17}$$

The homogeneity property (3.5) of the norm, i.e., $\|cf\| = |c|\,\|f\|$, is clearly valid in the present situation, but there is a slight problem with the positivity property (3.6). The integral of a function is not affected by altering the value of the function at a finite number of points, so if f is a function on $[a, b]$ that is zero except at a finite number of points, then $\|f\| = 0$ although f is not the zero function. For the class $PC(a, b)$ with which we are working, there are two ways out of this difficulty. One is to use the convention suggested by the Fourier convergence theorem, that is, to consider only functions $f \in PC(a, b)$ with the property that

$$f(x) = \tfrac{1}{2}\left[f(x-) + f(x+)\right] \quad \text{for all } x \in (a, b), \quad f(a) = f(a+), \quad f(b) = f(b-).$$

If $f \in PC(a, b)$ satisfies this condition and $f(x_0) \neq 0$, then $|f(x)| > 0$ on some interval containing x_0, and hence $\|f\| > 0$. (See Exercises 6 and 7.) The other is simply to agree to consider two functions as equal if they agree except at finitely many points. The reader can use whichever of these devices seems most comfortable; at any rate, we shall not worry any more about this matter.

The concepts of orthogonal and orthonormal sets of functions are defined just as for vectors in $\mathbf{C}^k$, and we can ask whether there is an analogue of Theorem 3.1. That is, given an orthonormal set $\{\phi_n\}$ in $PC(a, b)$, can we express an arbitrary $f \in PC(a,b)$ as $\sum\langle f,\phi_n\rangle\phi_n$? Here, for the first time, we have to confront the fact that the space $PC(a,b)$, unlike $\mathbf{C}^k$, is infinite-dimensional. This means, in particular, that we cannot tell whether the set $\{\phi_n\}$ contains "enough" functions to span the whole space just by counting how many functions are in it; after all, if one removes finitely many elements from an infinite set, there are still infinitely many left. It also means that the sum $\sum\langle f,\phi_n\rangle\phi_n$ will be an infinite series, so we have to worry about convergence. Hence there is some work to be done; but we can see that we are on the track of something very interesting by reconsidering the results of the previous chapter in the light of the ideas we have just developed.

Consider the functions

$$\phi_n(x) = (2\pi)^{-1/2}e^{inx}, \qquad n = 0, \pm1, \pm2, \ldots$$

We regard these functions as elements of the space $PC(-\pi,\pi)$; we then have

$$\langle\phi_m,\phi_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}\overline{e^{inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Thus $\{\phi_n\}_{-\infty}^{\infty}$ is an orthonormal set. Moreover, if the Fourier coefficients c_n of $f \in PC(-\pi,\pi)$ are defined as in Chapter 2, we have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{inx}}\,dx = (2\pi)^{-1/2}\langle f,\phi_n\rangle,$$

and hence

$$\sum_{-\infty}^{\infty} c_n e^{inx} = \sum_{-\infty}^{\infty}\Big[(2\pi)^{-1/2}\langle f,\phi_n\rangle\Big]\Big[(2\pi)^{1/2}\phi_n(x)\Big] = \sum_{-\infty}^{\infty}\langle f,\phi_n\rangle\phi_n(x).$$

Thus, the Fourier series of f is just its expansion with respect to the orthonormal set $\{\phi_n\}$, as one would expect from the discussion in §3.1!

Let us try this again for Fourier cosine series on the interval $[0,\pi]$. From the trigonometric identity

$$\cos a\cos b = \tfrac{1}{2}\Big[\cos(a+b) + \cos(a-b)\Big]$$

and the fact that

$$\int_0^{\pi}\cos kx\,dx = \begin{cases} k^{-1}\sin kx\big|_0^{\pi} = 0 & \text{for } k \neq 0, \\ x\big|_0^{\pi} = \pi & \text{for } k = 0, \end{cases}$$

we see that for $m, n \geq 0$,

$$\begin{aligned}\int_0^{\pi}\cos mx\cos nx\,dx &= \frac{1}{2}\int_0^{\pi}\Big[\cos(m+n)x + \cos(m-n)x\Big]dx \\ &= \begin{cases} \pi & \text{if } m = n = 0, \\ \tfrac{1}{2}\pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n. \end{cases}\end{aligned}$$

That is, if we define

$$\psi_0(x) = (1/\pi)^{1/2}, \qquad \psi_n(x) = (2/\pi)^{1/2} \cos nx \quad \text{for } n > 0,$$

then $\{\psi_n\}_0^\infty$ is an orthonormal set in $PC(0, \pi)$. Moreover, if the Fourier cosine coefficients a_n of $f \in PC(0, \pi)$ are defined as before,

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx = \begin{cases} 2(1/\pi)^{1/2} \langle f, \psi_0 \rangle & \text{for } n = 0, \\ (2/\pi)^{1/2} \langle f, \psi_n \rangle & \text{for } n > 0, \end{cases}$$

we have

$$\tfrac{1}{2} a_0 + \sum_0^\infty a_n \cos nx = \sum_0^\infty \langle f, \psi_n \rangle \psi_n(x).$$

The reader may verify that the trigonometric form of the Fourier series on $[-\pi, \pi]$ and the Fourier sine series on $[0, \pi]$ are also instances of expansions with respect to orthonormal sets.

Now, we have been a bit cavalier in this discussion. The reader will recall that we proved the validity of Fourier expansions only for piecewise smooth functions; for functions that are merely piecewise continuous there is no guarantee that the Fourier series will converge at any given point. What this means is that we need to take a closer look at questions of convergence in the context of the ideas from vector geometry that we are now using.

EXERCISES

1. Show that $\left\{(2/l)^{1/2} \sin(n - \frac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal set in $PC(0, l)$.
2. Show that $\left\{(2/l)^{1/2} \cos(n - \frac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal set in $PC(0, l)$.
3. Show that $f_0(x) = 1$ and $f_1(x) = x$ are orthogonal on $[-1, 1]$, and find constants a and b so that $f_2(x) = x^2 + ax + b$ is orthogonal to both f_0 and f_1 on $[-1, 1]$. What are the normalizations of f_0, f_1, and f_2?
4. Suppose $\{\phi_n\}$ is an orthonormal set in $PC(0, l)$, and let ϕ_n^+ and ϕ_n^- be the even and odd extensions of ϕ_n to $[-l, l]$. Show that $\left\{2^{-1/2}\phi_n^+\right\} \cup \left\{2^{-1/2}\phi_n^-\right\}$ is an orthonormal set in $PC(-l, l)$. (Hint: First show that $\left\{2^{-1/2}\phi_n^+\right\}$ and $\left\{2^{-1/2}\phi_n^-\right\}$ are orthonormal, and then that $\langle \phi_n^+, \phi_m^- \rangle = 0$ for all m, n.)
5. Let $\{\phi_n : n \geq 0\}$ be an orthonormal set in $PC(-l, l)$ such that ϕ_n is even when n is even and ϕ_n is odd when n is odd. Show that $\{\sqrt{2}\,\phi_n : n \text{ even}\}$ and $\{\sqrt{2}\,\phi_n : n \text{ odd}\}$ are orthonormal sets in $PC(0, l)$.
6. Suppose $f \in PC(a, b)$ and $f(x) = \frac{1}{2}\left[f(x-) + f(x+)\right]$ for all $x \in (a, b)$. Show that if $f(x_0) \neq 0$ for some $x_0 \in (a, b)$, then $f(x) \neq 0$ for all x in some interval containing x_0. (x_0 may be an endpoint of the interval.)
7. Show that if $f \in PC(a, b)$, $f \geq 0$, and $\int_a^b f(x)\, dx = 0$, then $f(x) = 0$ except perhaps at finitely many points. (Hint: By redefining f at its discontinuities, you can make f satisfy the conditions of Exercise 6.)

3.3 Convergence and completeness

If we visualize a k-dimensional vector $\mathbf{a}$ as the point in k-space with coordinates $(a_1, \ldots, a_k)$ rather than as an arrow, then $\|\mathbf{a} - \mathbf{b}\|$ is just the distance between the points $\mathbf{a}$ and $\mathbf{b}$ as defined by Euclidean geometry. Accordingly, the natural notion of convergence for vectors is that $\mathbf{a}_n \to \mathbf{a}$ if and only if $\|\mathbf{a}_n - \mathbf{a}\| \to 0$. This suggests a new definition of convergence for functions. Namely, if $\{f_n\}$ is a sequence of functions in $PC(a, b)$, we say that $f_n \to f$ **in norm** if $\|f_n - f\| \to 0$, that is,

$$f_n \to f \text{ in norm} \iff \int_a^b |f_n(x) - f(x)|^2 dx \to 0.$$

Convergence of f_n to f in norm thus means that the difference $f_n - f$ tends to zero in a suitable averaged sense over the interval $[a, b]$. It does not guarantee pointwise convergence, nor does pointwise convergence imply convergence in norm. For example, let $[a, b] = [0, 1]$. If we define

$$f_n(x) = 1 \quad \text{for } 0 \le x \le 1/n, \quad f_n(x) = 0 \quad \text{elsewhere},$$

then

$$\|f_n\|^2 = \int_0^1 |f_n(x)|^2 \, dx = \int_0^{1/n} dx = 1/n,$$

so $f_n \to 0$ in norm, but $f_n(0) = 1$ for all n, so f_n does not converge to zero pointwise. On the other hand, if

$$g_n(x) = n \quad \text{for } 0 < x < 1/n, \quad g_n(x) = 0 \quad \text{elsewhere},$$

then $g_n \to 0$ pointwise (in fact, $g_n(0) = 0$ for all n, and for any $x > 0$, $g_n(x) = 0$ for $n > |x|^{-1}$), but

$$\|g_n\|^2 = \int_0^1 |g_n(x)|^2 \, dx = \int_0^{1/n} n^2 \, dx = n,$$

so $g_n \not\to 0$ in norm. However, we have the following simple and useful result.

Theorem 3.2. *If $f_n \to f$ uniformly on $[a, b]$ $(-\infty < a < b < \infty)$, then $f_n \to f$ in norm.*

Proof: Uniform convergence means that there is a sequence $\{M_n\}$ of constants such that $|f_n(x) - f(x)| \le M_n$ for all $x \in [a, b]$ and $M_n \to 0$. But then

$$\|f_n - f\|^2 = \int_a^b |f_n(x) - f(x)|^2 \, dx \le \int_a^b M_n^2 \, dx = (b - a)M_n^2,$$

so $\|f_n - f\|$ tends to zero along with M_n. ∎

It should be mentioned that the norm and inner product are themselves continuous with respect to convergence in norm; that is, if $f_n \to f$ in norm, then

$$\|f_n\| \to \|f\|, \qquad \langle f_n, g\rangle \to \langle f, g\rangle \quad \text{and} \quad \langle g, f_n\rangle \to \langle g, f\rangle \quad \text{for all } g.$$

The verification is left to the reader (Exercises 1 and 2).

$PC(a,b)$ fails in one crucial respect to be a good infinite-dimensional analogue of Euclidean space, namely, it is not *complete.* This means, intuitively, that there are sequences that look like they ought to converge in norm, but which fail to have a limit in the space $PC(a,b)$. The formal definition is as follows. A sequence $\{\mathbf{a}_n\}_1^\infty$ of vectors (or functions or numbers) is called a **Cauchy sequence** if $\|\mathbf{a}_m - \mathbf{a}_n\| \to 0$ as $m, n \to \infty$, that is, if the terms in the sequence get closer and closer to each other as one goes further out in the sequence. A space S of vectors (or functions or numbers) is called **complete** if every Cauchy sequence in S has a limit in S. The real and complex number systems are complete, and it follows easily that the vector spaces $\mathbf{C}^k$ are complete for any k. The set R of rational numbers is not: if $\{r_n\}$ is a sequence of rational numbers with an irrational limit, such as the sequence of decimal approximations to π, then $\{r_n\}$ is Cauchy but has no limit in R.

One can see that $PC(a,b)$ is not complete by the following simple example. Take $[a,b] = [0,1]$, and let

$$f_n(x) = x^{-1/4} \quad \text{for } x > 1/n, \qquad f_n(x) = 0 \quad \text{for } x \le 1/n.$$

If $m > n$, $f_m(x) - f_n(x)$ equals $x^{-1/4}$ when $m^{-1} < x \le n^{-1}$ and equals 0 otherwise, so

$$\|f_m - f_n\|^2 = \int_{1/m}^{1/n} x^{-1/2}\, dx = 2x^{1/2}\Big|_{1/m}^{1/n} = 2(n^{-1/2} - m^{-1/2}),$$

which tends to zero as $m, n \to \infty$. Thus the sequence $\{f_n\}$ is Cauchy; but clearly its limit, either pointwise or in norm, is the function

$$f(x) = x^{-1/4} \quad \text{for } x > 0, \qquad f(0) = 0, \tag{3.18}$$

and this function does not belong to $PC(0,1)$ because it becomes unbounded as $x \to 0$.

It is easy enough to enlarge the space $PC(a,b)$ to include functions such as (3.18) with one or more infinite singularities in the interval $[a,b]$: One simply allows improper (but absolutely convergent) integrals in the definition of the inner product and the norm. But even this is not enough. One can construct Cauchy sequences $\{f_n\}$ in which f_n acquires more and more singularities as n increases, in such a way that the limit function f is everywhere discontinuous — and in particular, not Riemann integrable on any interval.

Fortunately, there is a more sophisticated theory of integration, the *Lebesgue integral,* which allows one to handle such highly irregular functions. The Lebesgue theory does require a very weak regularity condition called *measurability*, but this technicality need not concern us. All functions that arise in practice are measurable, and *all functions mentioned in the remainder of this book are tacitly assumed to be measurable.* For our present purposes, we do not need to know anything about the construction or detailed properties of the Lebesgue integral; all we need is a couple of definitions and a couple of facts that we shall quote without proof. Rudin [47] and Dym-McKean [19] contain brief expositions of Lebesgue integration that include most of the results we shall use; more extensive accounts of the theory can be found, for example, in Folland [25] and Wheeden-Zygmund [56].

We denote by $L^2(a,b)$ the space of **square-integrable** functions on $[a,b]$, that is, the set of all functions on $[a,b]$ whose squares are absolutely Lebesgue-integrable over $[a,b]$:

$$L^2(a,b) = \left\{ f : \int_a^b |f(x)|^2\,dx < \infty \right\}. \tag{3.19}$$

This space includes all functions for which the (possibly improper) Riemann integral $\int_a^b |f(x)|^2 dx$ converges, and one should think of it simply as the space of all functions f such that the region between the graph of $|f|^2$ and the x-axis has finite area. Since

$$st \le \tfrac{1}{2}(s^2 + t^2)$$

(because $s^2 + t^2 - 2st = (s-t)^2 \ge 0$) for any real numbers s and t, we have

$$|f(x)\overline{g(x)}| \le \tfrac{1}{2}\Big(|f(x)|^2 + |g(x)|^2\Big),$$

and thus if f and g are in $L^2(a,b)$, the integral

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

is absolutely convergent. Therefore, the definitions of the inner product and norm extend to the space $L^2(a,b)$, as do all their properties that we have discussed previously.

As in the space $PC(a,b)$, there is a slight problem with the positivity of the norm, as the condition $\int |f|^2 = 0$ does not imply that f vanishes identically but only that the $f = 0$ "almost everywhere." The precise interpretation of this phrase is as follows. A subset E of $\mathbf{R}$ is said to have **measure zero** if, for any $\epsilon > 0$, E can be covered by a sequence of open intervals whose total length is less than ϵ, that is, if there exist open intervals $I_1, I_2, \ldots$ of lengths $l_1, l_2, \ldots$ such that $E \subset \bigcup_1^\infty I_j$ and $\sum_1^\infty l_j < \epsilon$. (For example, any countable set has measure zero: If $E = \{x_1, x_2, \ldots\}$, let I_j be the interval of length $\epsilon/2^j$ centered at x_j.) A statement about real numbers that is true for all x except for those x in some set of measure zero is said to be true **almost everywhere**, or for **almost every** x.

It can be shown that if $f \in L^2(a,b)$, the norm of f is zero if and only if $f(x) = 0$ for almost every $x \in [a,b]$. Accordingly, we agree to regard two functions as equal if they are equal almost everywhere. This weakened notion of equality then validates the statement that $\|f\| = 0$ only when $f = 0$, and it turns out also to be appropriate in many other contexts. Moreover, if two *continuous* functions are equal almost everywhere then they are identically equal, so for continuous functions the ordinary notion of equality is entirely adequate.

The crucial properties of $L^2(a,b)$ that we shall need to state without proof are contained in the following theorem.

Theorem 3.3. (a) *$L^2(a,b)$ is complete with respect to convergence in norm.* (b) *For any $f \in L^2(a,b)$ there is a sequence f_n of continuous functions on $[a,b]$ such that $f_n \to f$ in norm. In fact, the functions f_n can be taken to be the restrictions to $[a,b]$ of functions on the line that possess derivatives of all orders at every point; moreover, the latter functions can be taken to be $(b-a)$-periodic or to vanish outside a bounded set.*

This theorem says that $L^2(a,b)$ is obtained by "filling in the holes" in the space $PC(a,b)$. The first assertion says that all the holes have been filled, and the second one says that nothing extra, beyond the completion of $PC(a,b)$, has been added in. For a proof, see Rudin [47], Theorems 11.38 and 11.42. We shall indicate how to prove the second assertion — that is, how to approximate arbitrary L^2 functions by smooth ones — in §7.1.

We are now ready to discuss the convergence of expansions with respect to orthonormal sets in $PC(a,b)$, or more generally in $L^2(a,b)$. The first step is to obtain the general form of Bessel's inequality, which is a straightforward generalization of the special case we proved in §2.1.

Bessel's Inequality. *If $\{\phi_n\}_1^\infty$ is an orthonormal set in $L^2(a,b)$ and $f \in L^2(a,b)$, then*

$$\sum_1^\infty |\langle f, \phi_n \rangle|^2 \le \|f\|^2. \tag{3.20}$$

Proof: Observe that

$$\Big\langle f, \langle f, \phi_n\rangle \phi_n \Big\rangle = \overline{\langle f, \phi_n\rangle} \langle f, \phi_n \rangle = |\langle f, \phi_n\rangle|^2$$

and that by the Pythagorean theorem,

$$\left\| \sum_1^N \langle f, \phi_n\rangle \phi_n \right\|^2 = \sum_1^N |\langle f, \phi_n\rangle|^2.$$

Hence, for any positive integer N, by Lemma 3.1,

$$\begin{aligned}0 &\leq \left\| f - \sum_1^N \langle f, \phi_n \rangle \phi_n \right\|^2 \\ &= \|f\|^2 - 2\,\mathrm{Re}\left\langle f, \sum_1^N \langle f, \phi_n \rangle \phi_n \right\rangle + \left\| \sum_1^N \langle f, \phi_n \rangle \phi_n \right\|^2 \\ &= \|f\|^2 - 2\sum_1^N |\langle f, \phi_n \rangle|^2 + \sum_1^N |\langle f, \phi_n \rangle|^2 \\ &= \|f\|^2 - \sum_1^N |\langle f, \phi_n \rangle|^2 .\end{aligned}$$

Letting $N \to \infty$, we obtain the desired result. ∎

We are now concerned with the following problem: given an orthonormal set $\{\phi_n\}_1^\infty$ in $L^2(a,b)$, is it true that

$$f = \sum_1^\infty \langle f, \phi_n \rangle \phi_n \tag{3.21}$$

for all $f \in L^2(a,b)$? First we assure ourselves that the series on the right actually makes sense.

Lemma 3.2. *If $f \in L^2(a,b)$ and $\{\phi_n\}$ is any orthonormal set in $L^2(a,b)$, then the series $\sum \langle f, \phi_n \rangle \phi_n$ converges in norm, and $\left\| \sum \langle f, \phi_n \rangle \phi_n \right\| \leq \|f\|$.*

Proof: Bessel's inequality guarantees that the series $\sum |\langle f, \phi_n \rangle|^2$ converges, so by the Pythagorean theorem,

$$\left\| \sum_m^n \langle f, \phi_n \rangle \phi_n \right\|^2 = \sum_m^n |\langle f, \phi_n \rangle|^2 \to 0 \quad \text{as } m, n \to \infty.$$

Thus the partial sums of the series $\sum \langle f, \phi_n \rangle \phi_n$ form a Cauchy sequence, and since $L^2(a,b)$ is complete, the series converges. Finally, another application of the Pythagorean theorem and Bessel's inequality gives

$$\begin{aligned}\left\| \sum_1^\infty \langle f, \phi_n \rangle \phi_n \right\|^2 = \lim_{N\to\infty} \left\| \sum_1^N \langle f, \phi_n \rangle \phi_n \right\|^2 &= \lim_{N\to\infty} \sum_1^N |\langle f, \phi_n \rangle|^2 \\ &= \sum_1^\infty |\langle f, \phi_n \rangle|^2 \leq \|f\|^2 .\end{aligned}$$ ∎

Now, an obvious necessary condition for (3.21) to hold for arbitrary f is that the orthonormal set $\{\phi_n\}$ is as large as possible, that is, that there is no nonzero f which is orthogonal to all the ϕ_n's. (If $\langle f, \phi_n \rangle = 0$ for all n, then (3.21) implies that $f = 0$.) Moreover, if (3.21) holds and the Pythagorean theorem extends to infinite sums of orthogonal vectors, Bessel's inequality (3.20) should actually be an equality. With these thoughts in mind, we arrive at the main theorem.

Theorem 3.4. *Let $\{\phi_n\}_1^\infty$ be an orthonormal set in $L^2(a,b)$. The following conditions are equivalent:*
(a) If $\langle f, \phi_n \rangle = 0$ for all n, then $f = 0$.
(b) For every $f \in L^2(a,b)$ we have $f = \sum_1^\infty \langle f, \phi_n \rangle \phi_n$, where the series converges in norm.
(c) For every $f \in L^2(a,b)$, we have **Parseval's equation**:

$$\|f\|^2 = \sum_1^\infty |\langle f, \phi_n \rangle|^2. \tag{3.22}$$

Proof: We shall show that (a) implies (b), that (b) implies (c), and that (c) implies (a).

(a) implies (b): Given $f \in L^2(a,b)$, the series $\sum \langle f, \phi_n \rangle \phi_n$ converges in norm, by Lemma 3.2. We can see that its sum is f by showing that the difference $g = f - \sum \langle f, \phi_n \rangle \phi_n$ is zero. But

$$\langle g, \phi_m \rangle = \langle f, \phi_m \rangle - \sum_{n=1}^\infty \langle f, \phi_n \rangle \langle \phi_n, \phi_m \rangle = \langle f, \phi_m \rangle - \langle f, \phi_m \rangle = 0$$

for all m. Hence, if (a) holds, $g = 0$.

(b) implies (c): If $f = \sum \langle f, \phi_n \rangle \phi_n$, then by the Pythagorean theorem,

$$\|f\|^2 = \lim_{N\to\infty} \left\| \sum_1^N \langle f, \phi_n \rangle \phi_n \right\|^2 = \lim_{N\to\infty} \sum_1^N |\langle f, \phi_n \rangle|^2 = \sum_1^\infty |\langle f, \phi_n \rangle|^2.$$

(c) implies (a): If (c) holds and $\langle f, \phi_n \rangle = 0$ for all n then $\|f\| = 0$, and therefore $f = 0$. ∎

An orthonormal set that possesses the properties (a)–(c) of Theorem 3.4 is called a **complete orthonormal set** or an **orthonormal basis** for $L^2(a,b)$. This usage of the word *complete* is different from the one discussed earlier in this section, but it is obviously appropriate in the present context. If $\{\phi_n\}$ is an orthonormal basis of $L^2(a,b)$ and $f \in L^2(a,b)$, the numbers $\langle f, \phi_n \rangle$ are called the (generalized) **Fourier coefficients** of f with respect to $\{\phi_n\}$, and the series $\sum \langle f, \phi_n \rangle \phi_n$ is called the (generalized) **Fourier series** of f.

Often it is more convenient not to require the elements of a basis to be unit vectors. Accordingly, suppose $\{\psi_n\}$ is an orthogonal set (and recall that, according to our definition of orthogonal set, this entails $\psi_n \neq 0$ for all n). Let $\phi_n = \|\psi_n\|^{-1}\psi_n$; then $\{\phi_n\}$ is an orthonormal set. We say that $\{\psi_n\}$ is a **complete orthogonal set** or an **orthogonal basis** if $\{\phi_n\}$ is an orthonormal basis. In this case the expansion formula for $f \in L^2(a,b)$ and the Parseval equation take the form

$$f = \sum \frac{\langle f, \psi_n \rangle}{\|\psi_n\|^2} \psi_n \qquad \|f\|^2 = \sum \frac{|\langle f, \psi_n \rangle|^2}{\|\psi_n\|^2}. \tag{3.23}$$

Now, what about the orthonormal sets derived from Fourier series that we discussed in §3.2? We have not yet proved that they are complete, for we derived the expansion formula $f = \sum \langle f, \phi_n \rangle \phi_n$ only when f was piecewise smooth, not for an arbitrary $f \in L^2(a,b)$. But there is actually very little work left to do.

Theorem 3.5. *The sets*

$$\left\{e^{inx}\right\}_{n=-\infty}^{\infty} \quad \text{and} \quad \left\{\cos nx\right\}_{n=0}^{\infty} \cup \left\{\sin nx\right\}_{n=1}^{\infty}$$

are orthogonal bases for $L^2(-\pi,\pi)$. *The sets*

$$\left\{\cos nx\right\}_{n=0}^{\infty} \quad \text{and} \quad \left\{\sin nx\right\}_{n=1}^{\infty}$$

are orthogonal bases for $L^2(0,\pi)$.

Proof: First consider the functions $\psi_n(x) = e^{inx}$. Suppose $f \in L^2(-\pi,\pi)$ and ϵ is a (small) positive number; we wish to show that the Nth partial sum of the Fourier series of f approximates f in norm to within ϵ if N is sufficiently large. By part (b) of Theorem 3.3, we can find a 2π-periodic function $\widetilde{f}$, possessing derivatives of all orders, such that $\|f - \widetilde{f}\| < \epsilon/3$. Let $c_n = (2\pi)^{-1}\langle f, \psi_n\rangle$ and $\widetilde{c}_n = (2\pi)^{-1}\langle \widetilde{f}, \psi_n\rangle$ be the Fourier coefficients of f and $\widetilde{f}$. By Theorem 2.5 of §2.3, we know that the Fourier series $\sum \widetilde{c}_n\psi_n$ converges uniformly to $\widetilde{f}$; hence, by Theorem 3.3, it converges to $\widetilde{f}$ in norm. Thus, if we take N sufficiently large, we have

$$\left\| \widetilde{f} - \sum_{-N}^{N} \widetilde{c}_n\psi_n \right\| < \frac{\epsilon}{3}.$$

Moreover, by the Pythagorean theorem and Bessel's inequality,

$$\begin{aligned} \left\| \sum_{-N}^{N} \widetilde{c}_n\psi_n - \sum_{-N}^{N} c_n\psi_n \right\|^2 &\le \sum_{-N}^{N} |\widetilde{c}_n - c_n|^2 \\ &\le \sum_{-\infty}^{\infty} |\widetilde{c}_n - c_n|^2 \le \|\widetilde{f} - f\|^2 < \left(\frac{\epsilon}{3}\right)^2. \end{aligned}$$

Thus, if we write

$$f - \sum_{-N}^{N} c_n\psi_n = \left(f - \widetilde{f}\right) + \left(\widetilde{f} - \sum_{-N}^{N} \widetilde{c}_n\psi_n\right) + \left(\sum_{-N}^{N} \widetilde{c}_n\psi_n - \sum_{-N}^{N} c_n\psi_n\right)$$

and use the triangle inequality, we see that

$$\left\| f - \sum_{-N}^{N} c_n\psi_n \right\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This proves the completeness of the set $\{\psi_n\} = \{e^{inx}\}$ in $L^2(-\pi,\pi)$, and the completeness of $\{\cos nx\} \cup \{\sin nx\}$ is essentially a restatement of the same result. The completeness of $\{\cos nx\}$ and $\{\sin nx\}$ in $L^2(0,\pi)$ is an easy corollary. (Just consider the even or odd extension of $f \in L^2(0,\pi)$ to $[-\pi,\pi]$.) ∎

The normalizing constants for the functions in Theorem 3.5 are, of course, $\sqrt{1/2\pi}$ for e^{inx}, $\sqrt{1/\pi}$ for $\cos nx$ and $\sin nx$ on $[-\pi, \pi]$ (except for $n = 0$), and $\sqrt{2/\pi}$ for $\cos nx$ and $\sin nx$ on $[0, \pi]$ (except for $n = 0$). With this in mind, one easily sees that the Parseval equation takes the form

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx = 2\pi \sum_{-\infty}^{\infty} |c_n|^2 = \frac{\pi}{2}|a_0|^2 + \pi \sum_{1}^{\infty} \left(|a_n|^2 + |b_n|^2\right), \qquad f \in L^2(-\pi, \pi),$$

where a_n, b_n, and c_n are the Fourier coefficients of f as defined in §2.1, and

$$\int_0^{\pi} |f(x)|^2\, dx = \frac{\pi}{4}|a_0|^2 + \frac{\pi}{2}\sum_1^{\infty} |a_n|^2 = \frac{\pi}{2}\sum_1^{\infty} |b_n|^2, \qquad f \in L^2(0, \pi),$$

where a_n and b_n are the Fourier cosine and sine coefficients of f as defined in §2.4. For example, if we consider the Fourier sine series of $f(x) = x$ on $[0, \pi]$ as derived in §2.1, we find that

$$\frac{\pi}{2}\sum_1^{\infty} \frac{4}{n^2} = \int_0^{\pi} x^2\, dx = \frac{\pi^3}{3}, \quad \text{or} \quad \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

a result which we derived by other means in Exercise 3, §2.3.

Let us sum up our theorems about the convergence of Fourier series. If f is a periodic function, then the Fourier series of f converges to f

(i) absolutely, uniformly, and in norm, if f is continuous and piecewise smooth;
(ii) pointwise and in norm, if f is piecewise smooth;
(iii) in norm, if $f \in L^2(a, b)$.

These results are sufficient for virtually all practical purposes. However, as we indicated in §2.6, there is more to be said on the subject. Here we shall just mention one more result that is a natural generalization of the theorems in this section. If $1 \le p < \infty$, we define $L^p(a, b)$ to be the space of Lebesgue-integrable functions f on $[a, b]$ such that

$$\int_a^b |f(x)|^p\, dx < \infty.$$

If $p > 1$, the Fourier series of any $f \in L^p(-\pi, \pi)$ converges to f in the "L^p norm," that is, if $\{c_n\}$ are the Fourier coefficients of f,

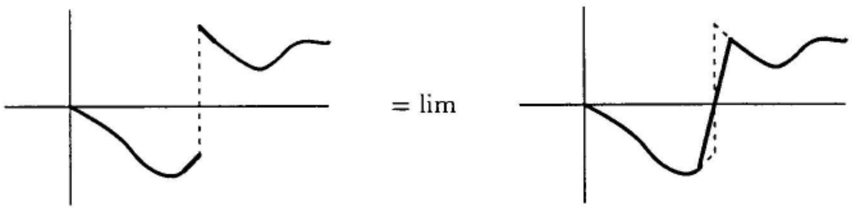
$$\int_a^b \left| \sum_{-N}^{N} c_n e^{inx} - f(x) \right|^p dx \to 0 \quad \text{as } N \to \infty.$$

However, this result is false for $p = 1$.

EXERCISES

1. Show that if $f_n \in L^2(a, b)$ and $f_n \to f$ in norm, then $\langle f_n, g\rangle \to \langle f, g\rangle$ for all $g \in L^2(a, b)$. (Hint: Apply the Cauchy-Schwarz inequality to $\langle f_n - f, g\rangle$.)
2. Show that $\big| \|f\| - \|g\| \big| \le \|f - g\|$. (Use the triangle inequality; consider the cases $\|f\| \ge \|g\|$ and $\|f\| \le \|g\|$ separately.) Deduce that if $f_n \to f$ in norm then $\|f_n\| \to \|f\|$.

3. Show directly that any $f \in PC(a,b)$ is the limit in norm of a sequence of continuous functions on $[a,b]$, by the argument suggested by the following picture.



4. Suppose $\{\phi_n\}$ is an orthonormal basis for $L^2(a,b)$. Suppose $c > 0$ and $d \in \mathbf{R}$, and let $\psi_n(x) = c^{1/2}\phi_n(cx+d)$. Show that $\{\psi_n\}$ is an orthonormal basis for $L^2(\frac{a-d}{c}, \frac{b-d}{c})$.
5. Finish the proof of Theorem 3.5. That is, from the completeness of $\{e^{inx}\}$ on $[-\pi,\pi]$, deduce the completeness of $\{\cos nx\} \cup \{\sin nx\}$ on $[-\pi,\pi]$ and the completeness of $\{\cos nx\}$ and $\{\sin nx\}$ on $[0,\pi]$.
6. Let $\phi_n(x) = (2/l)^{1/2}\sin(n-\frac{1}{2})(\pi x/l)$. In Exercise 1, §3.2, it was shown that $\{\phi_n\}_1^\infty$ is an orthonormal set in $L^2(0,l)$. Prove that it is actually a basis, via the following argument.
   a. Let $\psi_k(x) = l^{-1/2}\sin(k\pi x/2l)$. Show that $\{\psi_k\}_1^\infty$ is an orthonormal basis for $L^2(0,2l)$. (This follows from Theorem 3.5 and Exercise 4.)
   b. If $f \in L^2(0,l)$, extend f to $[0,2l]$ by making it symmetric about the line $x = l$, that is, define the extension $\tilde{f}$ by $\tilde{f}(x) = \tilde{f}(2l-x) = f(x)$ for $x \in [0,l]$. Show that $\langle \tilde{f}, \psi_{2n}\rangle = 0$ and $\langle \tilde{f}, \psi_{2n-1}\rangle = 2^{1/2}\langle f, \phi_n\rangle$.
   c. Conclude that if $\langle f, \phi_n\rangle = 0$ for all n, then $f = 0$.
7. Show that $\left\{(2/l)^{1/2}\cos(n-\frac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal basis for $L^2(0,l)$. (The argument is similar to that in Exercise 6, but this time you should extend f to be skew-symmetric about $x = l$, that is, $\tilde{f}(2l-x) = -\tilde{f}(x) = -f(x)$ for $x \in [0,l]$.)
8. Find the expansions of the functions $f(x) = 1$ and $g(x) = x$ on $[0,l]$ with respect to the orthonormal bases in Exercises 6 and 7.
9. Suppose $\{\phi_n\}$ is an orthonormal basis for $L^2(a,b)$. Show that for any $f, g \in L^2(a,b)$,

$$\langle f, g\rangle = \sum \langle f, \phi_n\rangle \overline{\langle g, \phi_n\rangle}.$$

   (Note that the case $f = g$ is Parseval's equation.)
10. Evaluate the following series by applying Parseval's equation to certain of the Fourier expansions in Table 1 of §2.1.

   a. $\displaystyle\sum_1^\infty \frac{1}{n^4}$ b. $\displaystyle\sum_1^\infty \frac{1}{(2n-1)^6}$ c. $\displaystyle\sum_1^\infty \frac{n^2}{(n^2+1)^2}$

   d. $\displaystyle\sum_1^\infty \frac{\sin^2 na}{n^4}$ $(0 < a < \pi)$

11. Suppose f is of class $C^{(1)}$, 2π-periodic, and real-valued. Show that f' is orthogonal to f in $L^2(-\pi,\pi)$ in two ways: (a) by expanding f in a Fourier series and using Exercise 9 and (b) directly from the fact that $2ff' = (f^2)'$.

3.4 More about L^2 spaces; the dominated convergence theorem

In this section we continue the general discussion of L^2 spaces and introduce an extremely useful criterion for the integral of a limit to equal the limit of the integrals.

Other types of L^2 spaces

The results of the previous section concerning $L^2(a,b)$ can be generalized in various ways, and we shall need some of these generalizations later on.

First, one can replace the element dx of linear measure on $[a,b]$ by a weighted element of measure, $w(x)\,dx$. To be precise, suppose w is a continuous function on $[a,b]$ such that $w(x) > 0$ for all $x \in [a,b]$; we call such a w a **weight function** on $[a,b]$. We can then define the "weighted L^2 space" $L^2_w(a,b)$ to be the set of all (Lebesgue measurable) functions on $[a,b]$ such that

$$\int_a^b |f(x)|^2 w(x)\,dx < \infty,$$

and we define an inner product and norm on $L^2_w(a,b)$ by

$$\langle f,g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,dx, \qquad \|f\|_w = \left(\int_a^b |f(x)|^2 w(x)\,dx\right)^{1/2}.$$

This inner product and norm still satisfy the fundamental conditions (3.3)–(3.6), so the theorems of §3.1 apply in this situation. So do Theorems 3.2, 3.3, and 3.4. w could also be allowed to have some singularities, as long as $\int_a^b w(x)\,dx < \infty$, or to vanish at a few points. (If w vanishes on a whole subinterval of $[a,b]$, one loses the strict positivity of the norm.)

Second, one can replace the bounded interval $[a,b]$ with a half-line or the whole line, or by a region in the plane or in a higher-dimensional space. That is, let D be a region in $\mathbf{R}^k$. (A "region" can be anything reasonable: an open set, or the closure of an open set, or indeed any Lebesgue measurable set. It does not have to be bounded, and indeed may be the whole space.) We define $L^2(D)$ to be the set of all functions f such that

$$\int_D |f(\mathbf{x})|^2\,d\mathbf{x} < \infty,$$

and we define the inner product and norm on $L^2(D)$ by

$$\langle f,g\rangle = \int_D f(\mathbf{x})\overline{g(\mathbf{x})}\,d\mathbf{x}, \qquad \|f\| = \left(\int_D |f(\mathbf{x})|^2\,d\mathbf{x}\right)^{1/2}.$$

Here $\int_D$ is a k-tuple integral, and $d\mathbf{x}$ is the element of Euclidean measure in k-space (length when $k = 1$, area when $k = 2$, volume when $k = 3$, etc.). If one is working only with Riemann integrals, one has to worry a bit about improper integrals when D is unbounded, but this problem is not serious. (The Lebesgue theory handles integrals over unbounded regions rather more smoothly.) Again, this inner product and norm satisfy (3.3)–(3.6), so the results of §3.1 are available, as is Theorem 3.4. However, the analogue of Theorem 3.2 is *false* when D is unbounded (or more precisely, when D has infinite measure), and a glance at its proof should show why. (See Exercise 6.) We shall state a result shortly that can be used in its place.

Theorem 3.3 also needs to be reformulated; here is one good version of it.

Theorem 3.6. *$L^2(D)$ is complete. If $f \in L^2(D)$, there is a sequence $\{f_n\}$ that converges to f in norm, such that each f_n is continuous on D and vanishes outside some bounded set. The f_n's can be taken to be restrictions to D of functions defined on all of $\mathbf{R}^k$ that have derivatives of all orders and vanish outside bounded sets.*

One can also modify $L^2(D)$ by throwing in a weight function, as before.

As a matter of fact, all one needs to develop the ideas of §3.1 are the following ingredients:

(i) a vector space $\mathscr{H}$, that is, a collection of objects that can be added to each other and multiplied by complex numbers, such that the usual laws of vector addition and scalar multiplication hold;

(ii) an inner product $\langle u, v\rangle$ on $\mathscr{H}$ and associated norm $\|u\| = \langle u, u\rangle^{1/2}$ that satisfy (3.3)–(3.6).

If, in addition, the space $\mathscr{H}$ is *complete* with respect to convergence in norm, it is called a **Hilbert space**. In this case, Bessel's inequality and Theorem 3.4 also hold. This general setup includes, but is not limited to, the spaces $\mathbf{C}^k$, $L^2(a, b)$, $L^2_w(a, b)$, and $L^2(D)$ discussed above.

Another example of a Hilbert space is the space l^2 of square-summable sequences. That is, the elements of l^2 are sequences $\{c_n\}_1^\infty$ of complex numbers such that $\sum_1^\infty |c_n|^2 < \infty$, and the inner product and norm are defined by

$$\Big\langle \{c_n\}, \{d_n\} \Big\rangle = \sum_1^\infty c_n \overline{d}_n, \qquad \Big\|\{c_n\}\Big\| = \left(\sum_1^\infty |c_n|^2\right)^{1/2}.$$

We have encountered this space before without mentioning it explicitly. Indeed, suppose $\{\phi_n\}_1^\infty$ is an orthonormal basis for $L^2(a, b)$. Then the mapping that takes an $f \in L^2(a, b)$ to its sequence of coefficients $\Big\{\langle f, \phi_n\rangle\Big\}$ sets up a one-to one correspondence between $L^2(a, b)$ and l^2 that is linear and (by Parseval's equation) norm-preserving. Such a mapping is called a **unitary operator**.

One further comment: We suggested thinking of functions $f \in L^2(a, b)$ as vectors whose components are the values $f(x)$, $x \in [a, b]$. The reader who knows about orders of infinity may be puzzled that there are uncountably many such "components," and yet the orthonormal bases we have displayed are countable sets. The explanation is that the elements of $L^2(a,b)$ are continuous functions or limits in norm of continuous functions, and the values of a continuous function are not completely independent of each other. For example, if f is continuous on $[a,b]$, then f is completely determined by its values at the rational points in $[a,b]$, of which there are only countably many.

The dominated convergence theorem

We now state one other result from the Lebesgue theory of integration that is of great utility even in the setting of Riemann integrable functions. It gives a general condition under which the integral of a limit is the limit of the integrals, and is an improvement on most of the theorems of this sort that one commonly encounters in calculus texts. We shall use it frequently throughout the rest of this book.

The Dominated Convergence Theorem. *Let D be a region in $\mathbf{R}^k$ $(k = 1, 2, 3, \ldots)$. Suppose g_n $(n = 1, 2, 3, \ldots)$, g, and ϕ are functions on D, such that*
(a) $\phi(\mathbf{x}) \geq 0$ and $\int_D \phi(\mathbf{x})\,d\mathbf{x} < \infty$,
(b) $|g_n(\mathbf{x})| \leq \phi(\mathbf{x})$ for all n and all $x \in D$,
(c) $g_n(\mathbf{x}) \to g(\mathbf{x})$ as $n \to \infty$ for all $\mathbf{x} \in D$.
Then $\int_D g_n(\mathbf{x})\,d\mathbf{x} \to \int_D g(\mathbf{x})\,d\mathbf{x}$.

The proof of this theorem is beyond the scope of this book (see Rudin [47], Folland [25], or Wheeden-Zygmund [56]), but the intuition behind it can be easily explained. If $g_n \to g$ pointwise, how can the relation $\int_D g_n \to \int_D g$ fail? Consider the following two examples, in which D is the real line:

$$f_n(x) = 1 \quad \text{for } n < x < n+1, \qquad f_n(x) = 0 \quad \text{otherwise.}$$
$$g_n(x) = n \quad \text{for } 0 < x < 1/n, \qquad g_n(x) = 0 \quad \text{otherwise.}$$

We have

$$\int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} g_n(x)\,dx = 1 \quad \text{for all } n,$$

but $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x) = 0$ for all x. The trouble is that as $n \to \infty$, the region under the graph of f_n moves out to infinity to the right, and the region under the graph of g_n moves out to infinity upwards, so in the limit there is nothing left. (See Figure 3.3.)

Now, the dominated convergence theorem essentially says that if this sort of bad behavior is eliminated, then the integral of the limit is the limit of the integrals. Hypothesis (a) says that the region under the graph of ϕ has finite area, and hypothesis (b) says that the graphs of $|g_n|$ are trapped inside this region, so they cannot leak out to infinity.

As a corollary, we obtain the following relation between pointwise convergence and convergence in norm.

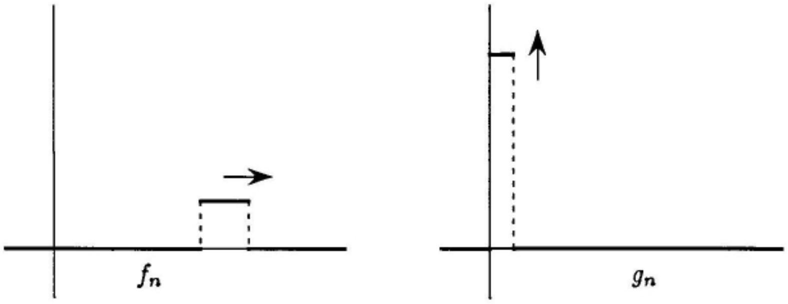


FIGURE 3.3. The examples f_n and g_n of sequences for which the integral of the limit is not the limit of the integral. The arrows indicate what happens as n increases.

Theorem 3.7. *Suppose $f_n \in L^2(D)$ for all n and $f_n \to f$ pointwise. If there exists $\psi \in L^2(D)$ such that $|f_n(\mathbf{x})| \le |\psi(\mathbf{x})|$ for all n and all $\mathbf{x} \in D$, then $f_n \to f$ in norm.*

Proof: We have $|f(\mathbf{x})| = \lim |f_n(\mathbf{x})| \le |\psi(\mathbf{x})|$, and hence

$$|f_n(\mathbf{x}) - f(\mathbf{x})|^2 \le \left(|f_n(\mathbf{x})| + |f(\mathbf{x})|\right)^2 \le |2\psi(\mathbf{x})|^2.$$

Therefore, we can apply the dominated convergence theorem, with $g_n = |f_n - f|^2$, $g = 0$, and $\phi = |2\psi|^2$, to conclude that

$$\|f_n - f\|^2 = \int_D |f_n(\mathbf{x}) - f(\mathbf{x})|^2 \, d\mathbf{x} \to 0. \qquad \blacksquare$$

Best approximations in L^2

If $\{\phi_n\}$ is an orthonormal basis for $L^2(D)$, where D is any interval in $\mathbf{R}$ or region in $\mathbf{R}^n$, we have $\sum \langle f, \phi_n \rangle \phi_n = f$ for all $f \in L^2(D)$. On the other hand, suppose $\{\phi_n\}$ is an orthonormal set in $L^2(D)$ that is not complete. If $f \in L^2(D)$, what significance can we attach to the series $\sum \langle f, \phi_n \rangle \phi_n$? We know that it converges by Lemma 3.2. In general its sum will not be f, but it is the unique *best approximation* to f in norm among all functions of the form $\sum c_n \phi_n$. (The latter sum converges in norm precisely when $\sum |c_n|^2 < \infty$, as the argument used to prove Lemma 3.2 shows.) We state this result as a theorem.

Theorem 3.8. *If $\{\phi_n\}$ is an orthonormal set in $L^2(D)$ and $f \in L^2(D)$, then*

$$\left\| f - \sum \langle f, \phi_n \rangle \phi_n \right\| \le \left\| f - \sum c_n \phi_n \right\|$$

for all choices of c_n with $\sum |c_n|^2 < \infty$. Equality holds only when $c_n = \langle f, \phi_n \rangle$ for all n.

Proof: We have

$$f - \sum c_n\phi_n = \left(f - \sum\langle f, \phi_n\rangle\phi_n\right) + \sum\left(\langle f, \phi_n\rangle - c_n\right)\phi_n.$$

Now, $f - \sum\langle f, \phi_n\rangle\phi_n$ is easily seen to be orthogonal to all ϕ_n; see the first part of the proof of Theorem 3.4. Hence, by the Pythagorean theorem (and a simple limiting argument, if there are infinitely many ϕ_n),

$$\left\| f - \sum c_n\phi_n \right\|^2 = \left\| f - \sum\langle f, \phi_n\rangle\phi_n \right\|^2 + \sum\left|\langle f, \phi_n\rangle - c_n\right|^2.$$

The last sum on the right is clearly nonnegative, and it is zero precisely when $c_n = \langle f, \phi_n\rangle$ for all n; this establishes the theorem. ∎

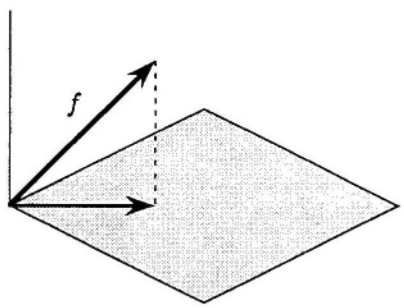


FIGURE 3.4. A vector f and its orthogonal projection onto a plane.

The pictorial intuition behind Theorem 3.8 is shown in Figure 3.4. The horizontal plane represents the space of functions (or vectors) of the form $\sum c_n\phi_n$; the sum $\sum\langle f, \phi_n\rangle\phi_n$ is the closest point to f in this plane, namely, the orthogonal projection of f onto the plane.

One situation in which Theorem 3.8 is particularly useful is when $\{\phi_n\}$ is simply a finite subset of an orthonormal basis.

Corollary 3.1. *Suppose $\{\phi_n\}_1^\infty$ is an orthonormal basis for $L^2(D)$. If $f \in L^2(D)$, the partial sum $\sum_1^N\langle f, \phi_n\rangle\phi_n$ of the series $\sum_1^\infty\langle f, \phi_n\rangle\phi_n$ is the best approximation in norm to f among all linear combinations of $\phi_1, \ldots, \phi_N$.*

EXERCISES

1. Show that $\left\{e^{2\pi i(mx+ny)}\right\}_{m,n=-\infty}^{\infty}$ is an orthonormal set in $L^2(D)$ where D is any square whose sides have length one and are parallel to the coordinate axes.
2. Find constants a, b, A, B, C such that $f_0(x) = 1$, $f_1(x) = ax + b$, and $f_2(x) = Ax^2 + Bx + C$ are an orthonormal set in $L_w^2(0, \infty)$ where $w(x) = e^{-x}$. (Hint: $\int_0^\infty x^n e^{-x}dx = n!$.)

3. Let D be the unit disc $\{x^2+y^2 \leq 1\}$, and let $f_n(x,y)=(x+iy)^n$. Show that $\{f_n\}_0^\infty$ is an orthogonal set in $L^2(D)$, and compute $\|f_n\|$ for all n. (Hint: In polar coordinates, $x+iy=re^{i\theta}$ and $dx\,dy=r\,dr\,d\theta$.)
4. Suppose $\{\phi_n\}$ is an orthonormal set in $L^2_w(D)$. Show that $\{w^{1/2}\phi_n\}$ is an orthonormal set in $L^2(D)$ (with respect to the weight function 1).
5. Suppose $f:[a,b]\to[c,d]$ and $f'(x)>0$ for $x\in[a,b]$. Show that if $\{\phi_n\}$ is an orthonormal basis for $L^2(c,d)$, then $\{\phi_n\circ f\}$ is an orthonormal basis for $L^2_w(a,b)$ where $w=f'$.
6. Find an example of a sequence $\{f_n\}$ in $L^2(0,\infty)$ such that $f_n\to 0$ uniformly but $f_n \not\to 0$ in norm.
7. What is the best approximation in norm to the function $f(x)=x$ on the interval $[0,\pi]$ among all functions of the form (a) $a_0+a_1\cos x+a_2\cos 2x$, (b) $b_1\sin x+b_2\sin 2x$, (c) $a\cos x+b\sin x$?

3.5 Regular Sturm-Liouville problems

In §1.3 we arrived at the orthogonal bases $\{\cos nx\}_0^\infty$ and $\{\sin nx\}_1^\infty$ for $L^2(0,\pi)$ by solving the boundary value problems

$$u''(x)+\lambda^2u(x)=0, \qquad u'(0)=u'(\pi)=0$$

and

$$u''(x)+\lambda^2u(x)=0, \qquad u(0)=u(\pi)=0.$$

We derived the orthogonal basis $\{e^{inx}\}_{-\infty}^\infty$ for $L^2(-\pi,\pi)$ by considering periodic functions, but we could also have found it by solving the boundary value problem

$$u''(x)+\lambda^2u(x)=0, \qquad u(-\pi)=u(\pi), \quad u'(-\pi)=u'(\pi).$$

In fact, there is a large class of boundary value problems on an interval $[a,b]$ that lead to orthogonal bases for $L^2(a,b)$. These problems are the subject of the present section.

First, a bit of conceptual background from finite-dimensional linear algebra. We recall that a linear transformation $T:\mathbf{C}^k\to\mathbf{C}^k$ is called *self-adjoint* or *Hermitian* if

$$\langle T\mathbf{a},\mathbf{b}\rangle=\langle\mathbf{a},T\mathbf{b}\rangle \quad \text{for all } \mathbf{a},\mathbf{b}\in\mathbf{C}^k.$$

(When T is described by a matrix (T_{ij}), this means that $T_{ji}=\overline{T_{ij}}$.) It is one of the basic results of linear algebra, known as the *spectral theorem* or the *principal axis theorem,* that whenever T is self-adjoint there is an orthonormal basis of $\mathbf{C}^k$ consisting of eigenvectors for T. What we are aiming for is an analogue of this theorem for differential operators acting on the space $L^2(a,b)$.

Suppose then that S and T are linear operators that are defined on certain subspaces $\mathscr{D}_S$ and $\mathscr{D}_T$ of $L^2(a,b)$ and map them into $L^2(a,b)$. We say that S and T are **adjoint** to each other (or that T is the adjoint of S, or vice versa) if

$$\langle S(f),g\rangle=\langle f,T(g)\rangle \quad \text{for all } f\in\mathscr{D}_S \text{ and } g\in\mathscr{D}_T.$$

S is called **self-adjoint** or **Hermitian** if

$$\langle S(f), g\rangle = \langle f, S(g)\rangle \quad \text{for all } f, g \in \mathscr{D}_S.$$

(These definitions will suffice for our purposes; in more advanced work one needs to be more careful about specifying the domains $\mathscr{D}_S$ and $\mathscr{D}_T$.)

Now suppose L is a second-order linear differential operator,

$$L(f) = rf'' + qf' + pf,$$

where r, q, and p are real functions of class $C^{(2)}$ on $[a, b]$. We shall assume that the leading coefficient r is nonvanishing on $[a, b]$, as the existence of "singular points" where $r = 0$ complicates the theory considerably. (Later we shall sometimes allow r to vanish at one or both endpoints.) For the time being, we take the domain of L to be the space of all twice continuously differentiable functions on $[a, b]$.

What is the adjoint of L? If we write out the integral defining $\langle L(f), g\rangle$, we can move the derivatives from f onto g by integration by parts, thus:

$$\int_a^b (rf'')\overline{g}\,dx = -\int_a^b f'(r\overline{g})'dx + rf'\overline{g}\Big|_a^b = \int_a^b f(r\overline{g})''dx + \Big[rf'\overline{g} - f(r\overline{g})'\Big]_a^b,$$

$$\int_a^b (qf')\overline{g}\,dx = -\int_a^b f(q\overline{g})'dx + qf\overline{g}\Big|_a^b.$$

We therefore have

$$\begin{aligned}\langle L(f), g\rangle &= \int_a^b (rf'' + qf' + pf)\overline{g}\,dx \\ &= \int_a^b f\Big[(r\overline{g})'' - (q\overline{g})' + p\overline{g}\Big]dx + \Big[rf'\overline{g} - f(r\overline{g})' + qf\overline{g}\Big]_a^b \qquad (3.24)\\ &= \langle f, L^*(g)\rangle + \Big[r(f'\overline{g} - f\overline{g}') + (q - r')f\overline{g}\Big]_a^b,\end{aligned}$$

where L^* is the **formal adjoint** of L defined by

$$L^*(g) = (rg)'' - (qg)' + pg = rg'' + (2r' - q)g' + (r'' - q' + p)g. \qquad (3.25)$$

(Here we have used the assumption that r, q, and p are real.) We say that L is **formally self-adjoint** if $L^* = L$. On comparing the coefficients of L^* with L, we see that this happens precisely when $2r' - q = q$ and $r'' - q' = 0$, that is, when $q = r'$. In this case, L has the form

$$L(f) = rf'' + r'f' + pf = (rf')' + pf, \qquad (3.26)$$

and moreover, the second boundary term at the end of (3.24) vanishes. We have therefore proved the following.

Lagrange's Indentity. *If L is formally self-adjoint,*

$$\langle L(f), g\rangle = \langle f, L(g)\rangle + \Big[r(f'\overline{g} - f\overline{g}')\Big]_a^b. \tag{3.27}$$

Evidently the discrepancy between formal and actual self-adjointness lies in the endpoint terms in (3.27). They can be eliminated by restricting L to a smaller domain, consisting of functions that satisfy suitable boundary conditions. More precisely, for a second-order operator L it is usually appropriate to impose two independent boundary conditions of the form

$$\begin{aligned} B_1(f) &= \alpha_1 f(a) + \alpha_1' f'(a) + \beta_1 f(b) + \beta_1' f'(b) = 0, \\ B_2(f) &= \alpha_2 f(a) + \alpha_2' f'(a) + \beta_2 f(b) + \beta_2' f'(b) = 0, \end{aligned} \tag{3.28}$$

where the α's and β's are constants. We say that the boundary conditions (3.28) are **self-adjoint** (relative to the operator L) if

$$\Big[r(f'\overline{g} - f\overline{g}')\Big]_a^b = 0 \quad \text{for all } f, g \text{ satisfying (3.28).}$$

Almost all the boundary conditions that arise in practice are of the form

$$\begin{gathered} \alpha f(a) + \alpha' f'(a) = 0, \qquad \beta f(b) + \beta' f'(b) = 0 \\ \Big(\alpha, \alpha', \beta, \beta' \in \mathbf{R}; \quad (\alpha, \alpha') \neq (0,0); \quad (\beta, \beta') \neq (0,0)\Big). \end{gathered} \tag{3.29}$$

Boundary conditions of the form (3.29) are called **separated**, since each one involves a condition at only one endpoint. Separated boundary conditions are always self-adjoint (relative to any operator L). In fact, if f and g both satisfy the boundary condition at a,

$$\alpha f(a) + \alpha' f'(a) = 0, \qquad \alpha g(a) + \alpha' g'(a) = 0, \tag{3.30}$$

then the expression $r(f'\overline{g} - f\overline{g}')$ vanishes at $x = a$; likewise at b. This is obvious when $\alpha' = 0$, in which case (3.30) becomes $f(a) = g(a) = 0$; on the other hand, if $\alpha' \neq 0$, we can rewrite (3.30) as

$$f'(a) = cf(a), \qquad g'(a) = cg(a) \qquad (c = -\alpha/\alpha'),$$

so that

$$r(a)[f'(a)\overline{g(a)} - f(a)\overline{g'(a)}] = cr(a)[f(a)\overline{g(a)} - f(a)\overline{g(a)}] = 0.$$

There is also one set of nonseparated boundary conditions that is commonly used, namely, the **periodic** boundary conditions

$$f(a) = f(b), \qquad f'(a) = f'(b). \tag{3.31}$$

These are self-adjoint relative to L provided that $r(a) = r(b)$, for then the endpoint evaluations at a and b in (3.27) cancel each other out.

Now we are ready to formulate the boundary value problems that lead to orthogonal bases for $L^2(a, b)$.

Definition. A **regular Sturm-Liouville problem** on the interval $[a, b]$ is specified by the following data:
(i) a formally self-adjoint differential operator L defined by $L(f) = (rf')' + pf$, where r, r', and p are real and continuous on $[a, b]$ and $r > 0$ on $[a, b]$;
(ii) a set of self-adjoint boundary conditions, $B_1(f) = 0$ and $B_2(f) = 0$, for the operator L;
(iii) a positive, continuous function w on $[a, b]$.
The object is *to find all solutions f of the boundary value problem*

$$L(f) + \lambda w f = 0, \quad \text{i.e., } \left[r(x)f'(x)\right]' + p(x)f(x) + \lambda w(x)f(x) = 0, \\ B_1(f) = B_2(f) = 0, \tag{3.32}$$

where λ is an arbitrary constant.

(A comment on condition (i): We have assumed from the outset that r does not vanish on $[a, b]$, so either $r > 0$ or $r < 0$. If $r < 0$, we simply replace r, p, and λ by $-r$, $-p$, and $-\lambda$, which leaves (3.32) unchanged.)

For most values of λ, the only solution of (3.32) is the trivial one, $f(x) \equiv 0$. If (3.32) has nontrivial solutions, λ is called an **eigenvalue** for the Sturm-Liouville problem, and the corresponding nontrivial solutions are called **eigenfunctions**. (This usage of the term *eigenvalue* is somewhat specialized. λ is an eigenvalue in the usual sense of the word, not of the operator L but rather of the operator M defined by $M(f) = -w^{-1}L(f)$.) If f and g satisfy (3.32), then so does any linear combination $c_1 f + c_2 g$ (this is just the superposition principle at work), so the set of all eigenfunctions for a given eigenvalue λ, together with the zero function, is a linear space called the **eigenspace** for λ.

We summarize the elementary properties of eigenvalues and eigenfunctions in the following theorem, which displays the importance of eigenfunctions from the point of view of orthogonal sets. We recall that if $w > 0$ is a weight function on $[a, b]$, the weighted inner product $\langle f, g\rangle_w$ is given by

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,dx = \langle wf, g\rangle = \langle f, wg\rangle. \tag{3.33}$$

Theorem 3.9. *Let a regular Sturm-Liouville problem (3.32) be given.*
(a) All eigenvalues are real.
(b) Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function w; that is, if f and g are eigenfunctions with eigenvalues λ and μ, $\lambda \neq \mu$, then

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,dx = 0.$$

(c) The eigenspace for any eigenvalue λ is at most 2-dimensional. If the boundary conditions are separated, it is always 1-dimensional.

Proof: (a) If λ is an eigenvalue, with eigenfunction f, then

$$\lambda\|f\|_w^2 = \langle \lambda w f, f\rangle = -\langle L(f), f\rangle = -\langle f, L(f)\rangle = \langle f, \lambda w f\rangle = \overline{\lambda}\langle f, wf\rangle = \overline{\lambda}\|f\|_w^2.$$

Here we have used (3.27) and (3.33) and the fact that f satisfies self-adjoint boundary conditions. Since $\|f\|_w^2 > 0$, we conclude that $\overline{\lambda} = \lambda$, that is, λ is real.

(b) Suppose $L(f) + \lambda w f = 0$ and $L(g) + \mu w g = 0$, where f and g are nonzero. We have just shown that λ and μ must be real, and by the same sort of argument,

$$\lambda\langle f, g\rangle_w = \langle \lambda w f, g\rangle = -\langle L(f), g\rangle = -\langle f, L(g)\rangle = \langle f, \mu w g\rangle = \mu\langle f, g\rangle_w.$$

Thus, if $\lambda \neq \mu$ we must have $\langle f, g\rangle_w = 0$.

(c) The fundamental existence theorem for ordinary differential equations (see Appendix 5) says that for any constants c_1 and c_2 there is a unique solution of $L(f) + \lambda w f = 0$ satisfying the initial conditions $f(a) = c_1$, $f'(a) = c_2$. That is, a solution is specified by two arbitrary constants c_1 and c_2, so the space of *all* solutions of $L(f) + \lambda w f = 0$ is 2-dimensional. Hence the space of solutions satisfying the given boundary conditions is *at most* 2-dimensional. Moreover, if the boundary conditions are separated, one of them has the form $\alpha f(a) + \alpha' f'(a) = 0$. This imposes the linear relation $\alpha c_1 + \alpha' c_2 = 0$ on the constants c_1 and c_2 and hence reduces the dimension of the solution space to one. (Of course the other boundary condition will usually reduce the dimension to zero; this is why there are nontrivial solutions only for certain special values of λ.) ∎

At this point it is not evident that a given Sturm-Liouville problem has any eigenfunctions at all. But, in fact, there are as many as anyone could wish for.

Theorem 3.10. *For every regular Sturm-Liouville problem*

$$(rf')' + pf + \lambda w f = 0, \qquad B_1(f) = B_2(f) = 0$$

on $[a, b]$, *there is an an orthonormal basis* $\{\phi_n\}_1^\infty$ *of* $L_w^2(a, b)$ *consisting of eigenfunctions. If* λ_n *is the eigenvalue for* ϕ_n, *then* $\lim_{n\to\infty} \lambda_n = +\infty$. *Moreover, if* f *is of class* $C^{(2)}$ *on* $[a, b]$ *and satisfies the boundary conditions* $B_1(f) = B_2(f) = 0$, *then the series* $\sum\langle f, \phi_n\rangle\phi_n$ *converges uniformly to* f.

In more detail, the content of Theorem 3.10 is as follows. By Theorem 3.9(c), for each eigenvalue λ there are either one or two independent eigenfunctions. In the latter case we can choose the two eigenfunctions to be orthogonal to each other with respect to the weight w. (If $\langle f_1, f_2\rangle_w \neq 0$, we can replace f_2 by $\tilde{f}_2 = f_2 - cf_1$ where c is chosen to make $\langle f_1, \tilde{f}_2\rangle = 0$.) If we put all these eigenfunctions together, by Theorem 3.9(b) we obtain an orthogonal set; and Theorem 3.10 says that this set is actually a basis. This implies, in particular, that the set of eigenvalues is countably infinite.

We shall take Theorem 3.10 on faith for the present, but we shall prove it in the case of separated boundary conditions in §10.3. A proof of the general case, as well as its generalization to higher-order differential equations, can be found in Naimark [40], Chapter II.

Example. Consider the problem

$$f'' + \lambda f = 0, \qquad f'(0) = \alpha f(0), \quad f'(l) = \beta f(l). \tag{3.34}$$

First let us dispose of the case $\lambda = 0$. The general solution of $f'' = 0$ is $f(x) = c_1 + c_2 x$. The boundary condition at 0 says that $c_2 = \alpha c_1$, and the boundary condition at l says that $c_2 = \beta(c_1 + c_2 l)$. The only solution of this pair of equations is $c_1 = c_2 = 0$ unless $\beta = \alpha/(1+l\alpha)$, in which case we may take $c_1 = 1$ and $c_2 = \alpha$.

Now for $\lambda \neq 0$, let us set $\lambda = \nu^2$, where ν is positive real or positive imaginary according as $\lambda > 0$ or $\lambda < 0$. (By Theorem 3.9(a), we need only consider real λ.) The general solution of the differential equation $f'' + \lambda f = 0$ is

$$f(x) = c_1 \cos \nu x + c_2 \sin \nu x \qquad (\lambda = \nu^2).$$

Since $f(0) = c_1$ and $f'(0) = \nu c_2$, the boundary condition at 0 says that $c_2 = (\alpha/\nu)c_1$. Since a constant multiple of a solution is a solution, we may choose $c_1 = \nu$, $c_2 = \alpha$, so that

$$f(x) = \nu \cos \nu x + \alpha \sin \nu x. \tag{3.35}$$

Now the boundary condition at l says that

$$-\nu^2 \sin \nu l + \alpha\nu \cos \nu l = \beta(\nu \cos \nu l + \alpha \sin \nu l),$$

or

$$(\alpha - \beta)\nu \cos \nu l = (\alpha\beta + \nu^2) \sin \nu l,$$

or finally

$$\tan \nu l = \frac{(\alpha - \beta)\nu}{\alpha\beta + \nu^2}. \tag{3.36}$$

For the case of imaginary ν (i.e., $\lambda < 0$) we set $\nu = i\mu$ and use the fact that $\tan ix = i \tanh x$ to rewrite (3.36) as

$$\tanh \mu l = \frac{(\alpha - \beta)\mu}{\alpha\beta - \mu^2}. \tag{3.37}$$

In both cases we need only consider positive values of ν and μ, since the actual eigenvalue is ν^2 or $-\mu^2$.

If ν satisfies (3.36), then the function f defined by (3.35) is an eigenfunction for the problem (3.34). In general it is not normalized, but finding the normalization is a simple matter of calculus, and the equation (3.36) can often be used to simplify the result. As an illustration, let us work out the case $\beta = -\alpha$. (Other cases are considered in Exercises 5 and 6.) If f is given by (3.35), then

$$\begin{aligned}
\|f\|^2 &= \int_0^l (\nu^2 \cos^2 \nu x + 2\alpha\nu \sin \nu x \cos \nu x + \alpha^2 \sin^2 \nu x)\, dx \\
&= \Big[\tfrac{1}{2}\nu^2(x + \nu^{-1} \cos \nu x \sin \nu x) + \alpha \sin^2 \nu x + \tfrac{1}{2}\alpha^2(x - \nu^{-1} \cos \nu x \sin \nu x)\Big]_0^l \\
&= \tfrac{1}{2}(\nu^2 + \alpha^2)l + \frac{(\nu^2 - \alpha^2)}{2\nu} \cos \nu l \sin \nu l + \alpha \sin^2 \nu l.
\end{aligned}$$

But if $\beta = -\alpha$, (3.36) gives

$$\frac{(\nu^2 - \alpha^2)}{2\nu} = \frac{\alpha}{\tan \nu l} = \frac{\alpha \cos \nu l}{\sin \nu l},$$

so

$$\|f\|^2 = \tfrac{1}{2}(\nu^2 + \alpha^2)l + \alpha(\cos^2 \nu l + \sin^2 \nu l) = \tfrac{1}{2}(\nu^2 + \alpha^2)l + \alpha. \qquad (3.38)$$

There is no way to describe the values of ν and μ that solve the transcendental equations (3.36) and (3.37) in closed form (except when $\alpha = \beta$), but it is easy to find them graphically. Namely, they are the values at which the curves $y = \tan \nu l$ and $y = (\alpha - \beta)\nu/(\alpha\beta + \nu^2)$ in the νy-plane, or $y = \tanh \mu l$ and $y = (\alpha - \beta)\mu/(\alpha\beta - \mu^2)$ in the μy-plane, intersect. The relative configuration of these curves depends on α and β; we shall display a couple of representative cases here and let the reader work out some others as exercises.

Case I. $\alpha = 1$, $\beta = -1$, $l = \pi$. Here the situation is as depicted in Figure 3.5. There is an infinite sequence of positive solutions to (3.36), say $\nu_1 < \nu_2 < \cdots$, and ν_n is approximately $n - 1$ when n is large. There are no positive solutions to (3.37). Hence, there is an infinite sequence of positive eigenvalues $\lambda_n = \nu_n^2$ for (3.34), with $\lambda_n \approx (n-1)^2$ for n large, and no negative eigenvalues. (Zero is not an eigenvalue since $-1 \neq 1/(1+\pi)$.) The (unnormalized) eigenfunctions are given by (3.35):

$$f_n(x) = \nu_n \cos \nu_n x + \sin \nu_n x.$$

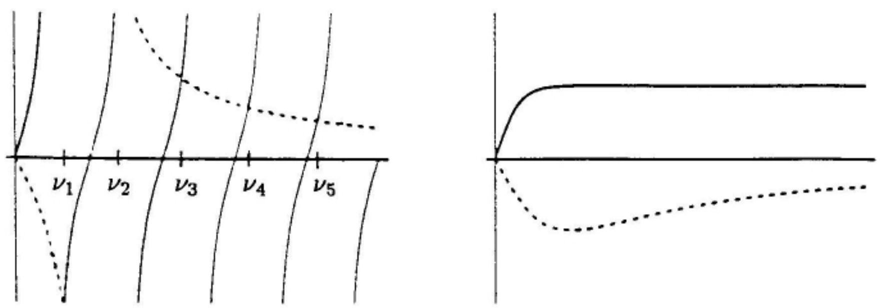


FIGURE 3.5. Left: the graphs of $\tan \pi\nu$ (solid) and $2\nu/(\nu^2 - 1)$ (dashed); the numbers ν_n are the values of ν at which the graphs intersect. Right: the graphs of $\tanh \pi\mu$ (solid) and $-2\mu/(\mu^2 + 1)$ (dashed).

Case II. $\alpha = 1$, $\beta = 4$, $l = \pi$. Here the situation is as depicted in Figure 3.6. Again there is an infinite sequence $\{\nu_n\}_1^\infty$ of positive solutions to (3.36), this time with $\nu_n \approx n$ for large n; and zero is not an eigenvalue of (3.34) since $4 \neq 1/(1+\pi)$. But now there is also one positive solution μ_0 to (3.37). Hence, there is an infinite sequence of positive eigenvalues $\lambda_n = \nu_n^2$ for (3.34) and one negative eigenvalue $\lambda_0 = -\mu_0^2$. The (unnormalized) eigenfunction for $\lambda_n = \nu_n^2$ is

$$f_n(x) = \nu_n \cos \nu_n x + \sin \nu_n x,$$

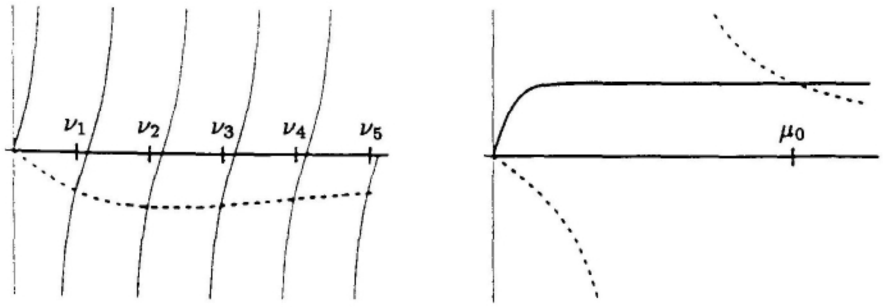


FIGURE 3.6. Left: the graphs of $\tan \pi\nu$ (solid) and $-3\nu/(\nu^2+4)$ (dashed). Right: the graphs of $\tanh \pi\mu$ (solid) and $3\mu/(\mu^2-4)$ (dashed). The numbers ν_n and μ_0 are the values of ν and μ at which the graphs intersect.

and the eigenfunction for $\lambda_0 = -\mu_0^2$ is

$$f_0(x) = \mu_0 \cosh \mu_0 x + \sinh \mu_0 x.$$

EXERCISES

1. Under what condition on the constants c and c' are the boundary conditions $f(b) = cf(a)$ and $f'(b) = c'f'(a)$ self-adjoint for the operator $L(f) = (rf')' + pf$ on $[a, b]$? (Assume as usual that r and p are real.)
2. Show that the problem (3.34) has no negative eigenvalues if $\alpha > 0 > \beta$ and exactly one negative eigenvalue if $\beta > \alpha > 0$ or $0 > \beta > \alpha$.
3. Find the eigenvalues and normalized eigenfunctions for the problem $f'' + \lambda f = 0$, $f(0) = 0$, $f'(l) = 0$ on $[0, l]$. (Cf. Exercise 6, §3.3.)
4. Find the eigenvalues and normalized eigenfunctions for the problem $f'' + \lambda f = 0$, $f'(0) = 0$, $f(l) = 0$ on $[0, l]$. (Cf. Exercise 7, §3.3.)
5. Find the normalized eigenfunctions for the problem (3.34) in the case $\alpha = 0$. (The answer is a bit different in the cases $\beta > 0$, $\beta = 0$, and $\beta < 0$.)
6. Find the normalized eigenfunctions for the problem (3.34) in the case $\beta = 0$. (Hint: The change of variable $x \to l - x$ essentially reduces this to Exercise 5.)
7. Find the eigenvalues and normalized eigenfunctions for the problem $f'' + \lambda f = 0$, $f(0) = 0$, $f'(1) = -f(1)$.
8. The Sturm-Liouville theory can be generalized to higher-order equations. As an example, consider the operator $L(f) = f^{(4)}$ on the interval $[0, l]$.
   a. Prove the analogue of Lagrange's identity for L:

$$\int_0^l \left[f^{(4)}(x)\overline{g}(x) - f(x)\overline{g}^{(4)}(x)\right] dx = \left[f'''\overline{g} - f\overline{g}''' + f'\overline{g}'' - f''\overline{g}'\right]_0^l. \quad (*)$$

   b. For the fourth-order equation $L(f) - \lambda f = 0$ one needs four boundary conditions involving f, f', f'', and f'''. Such a set of boundary conditions is called self-adjoint for L if the right side of $(*)$ vanishes whenever f and g both satisfy the conditions. Show that one obtains a self-adjoint set of boundary conditions by imposing any of the following pairs of conditions at $x = 0$ and any one of them at $x = l$:

$$f = f' = 0, \qquad f = f'' = 0, \qquad f' = f''' = 0, \qquad f'' = f''' = 0.$$

c. Show that the eigenvalues for the equation $L(f) - \lambda f = 0$, subject to any self-adjoint set of boundary conditions, are all real, and that eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(0, l)$.

d. One can show that the analogue of Theorem 3.10 holds here, i.e., there is an orthonormal basis of eigenfunctions. For example, consider the boundary conditions $f(0) = f''(0) = 0$, $f(l) = f''(l) = 0$. Show that $f_n(x) = \sin(n\pi x/l)$ is an eigenfunction. What is its eigenvalue? Why can you guarantee immediately that there are no other independent eigenfunctions?

9. Suppose p, q, and r are real functions of class $C^{(2)}$ and that $r > 0$. The differential equation $rf'' + qf' + pf + \lambda f = 0$ can be written in the form $L(f) + \lambda w f = 0$ where w is an arbitrary positive function and $L(f) = wrf'' + wqf' + wpf$. Show that w can be always be chosen so that L is formally self-adjoint.

The following two problems use the fact that the general solution of the Euler equation

$$x^2 f''(x) + axf'(x) + bf(x) = 0 \qquad (x > 0)$$

is $c_1 x^{r_1} + c_2 x^{r_2}$ where r_1 and r_2 are the zeros of the polynomial $r(r-1) + ar + b$. (If the two zeros coincide, the general solution is $c_1 x^{r_1} + c_2 x^{r_1} \log x$.) In case r_1 and r_2 are complex, it is useful to recall that $x^{is} = e^{is \log x}$.

10. Find the eigenvalues and normalized eigenfunctions for the problem

$$(xf')' + \lambda x^{-1} f = 0, \qquad f(1) = f(b) = 0 \qquad (b > 1).$$

Expand the function $g(x) = 1$ in terms of these eigenfunctions. (Hint: in computing integrals, make the substitution $y = \log x$. Orthonormality here is with respect to the weight $w(x) = x^{-1}$.)

11. Find the eigenvalues and normalized eigenfunctions for the problem

$$(x^2 f')' + \lambda f = 0, \qquad f(1) = f(b) = 0 \qquad (b > 1).$$

12. Consider the Sturm-Liouville problem

$$(rf')' + pf + \lambda f = 0, \qquad f(a) = f(b) = 0. \qquad (**)$$

a. Show that if f satisfies $(**)$, then

$$\lambda \int_a^b |f|^2 \, dx = \int_a^b r|f'|^2 \, dx - \int_a^b p|f|^2 \, dx.$$

(Hint: Use the fact that $\lambda f = -(rf')' - pf$ and integrate by parts.)

b. Deduce that if $p(x) \leq C$ for all x, then all the eigenvalues λ of (**) satisfy $\lambda \geq -C$.
c. Show that the conclusion of (b) still holds if the boundary conditions $f(a) = f(b) = 0$ are replaced by $f'(a) - \alpha f(a) = f'(b) - \beta f(b) = 0$ where $\alpha \leq 0$ and $\beta \geq 0$. (Hint: The analogue of part (a) in this situation is

$$\lambda \int_a^b |f|^2 \, dx = \int_a^b r|f'|^2 \, dx - \int_a^b p|f|^2 \, dx + \beta r(b)|f(b)|^2 - \alpha r(a)|f(a)|^2.)$$

3.6 Singular Sturm-Liouville problems

In §3.5 we considered the differential equation

$$rf'' + r'f' + pf + \lambda w f = 0 \tag{3.39}$$

on a closed, bounded interval $[a,b]$, in which r, r', p, and w were assumed continuous on $[a,b]$ and r and w were assumed strictly positive on $[a,b]$. However, it often turns out in practice that one or more of these assumptions must be weakened, leading to the so-called **singular Sturm-Liouville problems**. Specifically, we allow the following modifications of the basic setup:

(i) The leading coefficient r may vanish at one or both endpoints of $[a,b]$. In addition, the weight w may vanish or tend to infinity at one or both endpoints, and the function $|p|$ may tend to infinity at one or both endpoints.
(ii) The interval $[a,b]$ may be unbounded, that is, $a = -\infty$ and/or $b = \infty$.

There is an extensive theory of these more general boundary value problems, but it is beyond the scope of this book. (Complete treatments can be found in Dunford-Schwartz [18] and Naimark [40]; see also Titchmarsh [52].) We shall merely sketch a few of the main features here, and we shall discuss specific examples in Chapters 5 and 6 and Sections 7.4 and 10.4.

The first problem is to decide what sort of boundary conditions to impose. Since we wish to use the machinery of inner products and orthogonality, we wish to use only solutions of (3.39) that are square-integrable. Now, in the regular case, all solutions of (3.39) are continuous on $[a,b]$ and hence belong to $L^2_w(a,b)$. However, under condition (i), the solutions to (3.39) may fail to be square-integrable because they blow up at one or both endpoints; whereas under condition (ii), solutions may fail to be square-integrable because they do not decay at infinity. Thus, we distinguish two cases concerning the behavior of solutions at each endpoint; to be definite, we consider the endpoint a.

Case I. All solutions of (3.39) belong to $L^2_w(a,c)$ for $a < c < b$. (It turns out that if this condition is satisfied for one value of λ, then it is satisfied for all values of λ.) In this case, we impose a boundary condition at a. In some cases it may be of the form $\alpha f(a) + \alpha' f'(a) = 0$, as before, but it may also be a condition on the limiting behavior of f and f' at a — for example, the condition that $f(x)$ should remain bounded as $x \to a$.

Case II. Not all solutions of (3.39) belong to $L^2_w(a,c)$. In this case we impose no boundary condition at a beyond the one that automatically comes with the problem, namely, that the solution should belong to $L^2_w(a,b)$.

In any event, we require the boundary conditions to be self-adjoint, i.e., if f and g satisfy the boundary conditions then the boundary term in Lagrange's identity should vanish. Precisely, since f and g may have singularities at a and b, or a and/or b may be infinite, this requirement should be formulated as

$$\lim_{\delta,\epsilon\to 0}\Big[r(f'\overline{g} - f\overline{g}')\Big]_{a+\delta}^{b-\epsilon} = 0. \tag{3.40}$$

(3.40) implies that

$$\langle L(f), g\rangle = \langle f, L(g)\rangle \quad \text{where} \quad L(f) = (rf')' + pf,$$

for any smooth functions f and g that satisfy the boundary conditions, and once this equation is established, the proof of Theorem 3.9 goes through without change. Therefore, the eigenvalues are all real and the eigenfunctions with distinct eigenvalues are orthogonal to each other.

However, the situation with Theorem 3.10 is different: in general, *there is no guarantee that there will be enough eigenfunctions to make an orthonormal basis.* Sometimes there are, sometimes there aren't. In the latter case, it is still possible to expand arbitrary functions in $L^2_w(a,b)$ in terms of solutions of the differential equation (3.39) that satisfy the given boundary conditions, but the expansion will involve an integral rather than (or in addition to) an infinite series.

For example, consider the differential equation

$$f'' + \lambda f = 0 \quad \text{on} \quad (-\infty,\infty).$$

The general solution is

$$c_1\cos\nu x + c_2\sin\nu x \quad \text{or} \quad c_1e^{i\nu x} + c_2e^{-i\nu x} \quad (\lambda = \nu^2).$$

None of these functions, for any value of λ, belongs to $L^2(-\infty,\infty)$, except for the trivial case $c_1 = c_2 = 0$. However, any $f \in L^2(-\infty,\infty)$ can be written as a "continuous superposition" (i.e., integral) of the functions $e^{i\nu x}$ as ν ranges over all real numbers, by means of the Fourier transform. This is the subject of Chapter 7.

CHAPTER 4
SOME BOUNDARY VALUE PROBLEMS

This chapter is devoted to the solution of various boundary value problems by the techniques we have developed so far, namely,

(i) separation of variables,
(ii) the superposition principle, and
(iii) expansion of functions in series of eigenfunctions.

This subject was begun in §2.5. All the major ideas we need are already in place, and it is just a question of learning how to combine them efficiently and developing a feeling for the connection between the mathematics and the physics. In the first section we discuss a few useful general techniques; the remainder of the chapter is largely devoted to working out a variety of examples.

Our methods generally lead to solutions in the form of infinite series. In this chapter we shall not worry much about technical questions of convergence, termwise differentiation, and such things. In some cases, one can verify that the series converge in a sufficiently strong sense to justify all the formal manipulations according to the principles of classical analysis; even when this is not the case, one can usually establish the validity of the solution by interpreting things properly — for example, by abandoning pointwise convergence in favor of norm convergence or the notion of weak convergence that we shall develop in Chapter 9. These issues were discussed in some detail in §2.5 for the boundary value problems solved there, and similar remarks apply to the problems considered in this chapter. At any rate, our concern here is with finding the solutions rather than with a rigorous justification of the calculations.

We shall also not worry about questions of uniqueness. That is, our methods will produce *one* solution to the boundary value problem, and we shall not try to prove rigorously that it is the *only* solution. In general, a problem that is properly posed from a physical point of view will indeed have a unique solution; or at least any non-uniqueness will be easily visible in the physical setup. (See John [33] or Folland [24] for further discussion of these matters.)

We shall point out here and there how Sturm-Liouville problems of a rather general sort turn up in applications. However, when we perform specific calculations, we must limit ourselves to the differential equations that we can solve explicitly — and at this point, this means mainly the equation $f'' + \lambda f = 0$ or its close relative $x^2 f'' + 2xf' + \lambda f = 0$ (discussed in §4.3). We shall solve some problems involving more complicated equations in Chapters 5 and 6.

4.1 Some useful techniques

We begin this chapter by discussing the sort of problems we shall be considering and assembling a bag of tricks for them. To put the discussion on a concrete level, let us think of the boundary value problems for the heat and wave equations that we solved in §2.5. In these problems we were solving a homogeneous linear differential equation $L(u) = 0$ (either the heat or the wave equation) for a function $u(x,t)$ on the region $a < x < b$, $t > 0$. We imposed some homogeneous linear boundary conditions on u at $x = a$ and $x = b$, and some linear initial conditions on u at $t = 0$. Let us write the boundary conditions as $B(u) = 0$ and the initial conditions as $I(u) = h(x)$, with the understanding that each of these single equations may stand for several equations grouped together. (For example, for the vibrating string problem, "$B(u) = 0$" stands for "$u(0,t) = 0$ and $u(l,t) = 0$," and "$I(u) = h(x)$" stands for "$u(x,0) = h_1(x)$ and $u_t(x,0) = h_2(x)$.") Thus the boundary value problem has the form

$$L(u) = 0, \qquad B(u) = 0, \qquad I(u) = h(x). \tag{4.1}$$

The technique for solving (4.1) was to use separation of variables to produce an infinite family of functions $u(x,t) = \sum c_n \phi_n(t)\psi_n(x)$ that satisfy $L(u) = 0$ and $B(u) = 0$, and then to choose the constants c_n appropriately to obtain $I(u) = h(x)$.

In the examples we considered in §2.5, the boundary conditions were such as to lead to Fourier sine or cosine series. In this chapter we shall consider other homogeneous boundary conditions. These will yield other Sturm-Liouville problems and hence lead to infinite series involving the eigenfunctions for these problems. The particular eigenfunctions will differ from problem to problem, but the method of solution is the same in all cases.

We shall also generalize (4.1) by considering inhomogeneous equations and inhomogeneous boundary conditions:

$$L(u) = F(x,t), \qquad B(u) = g(t), \qquad I(u) = h(x). \tag{4.2}$$

There are several techniques for reducing such problems to more manageable ones. We now discuss these techniques on the general level; specific examples will be found in subsequent sections. The reader may find it helpful to read this material in conjunction with the examples, rather than trying to absorb it completely before reading further.

Technique 1: Use the superposition principle to deal with inhomogeneous terms one at a time.

In problem (4.2) there are three inhomogeneous terms: F, g, and h. Suppose we can solve the three problems obtained by replacing all but one of these functions by zero:

$$L(u) = 0, \qquad B(u) = 0, \qquad I(u) = h(x), \tag{4.3}$$
$$L(u) = 0, \qquad B(u) = g(t), \qquad I(u) = 0, \tag{4.4}$$
$$L(u) = F(x,t), \qquad B(u) = 0, \qquad I(u) = 0. \tag{4.5}$$

If u_1, u_2, and u_3 are the solutions to (4.3), (4.4), and (4.5), respectively, then $u = u_1 + u_2 + u_3$ will be the solution of (4.2). In particular, (4.3) is just (4.1), which we already know how to deal with, so it suffices to solve (4.4) and (4.5).

We remark that this method can sometimes be used to break down the problem still further. For example, if we are working on the interval $a < x < b$, the boundary condition $B(u) = g(t)$ generally stands for two conditions, one at $x = a$ and one at $x = b$, say $B_a(u) = g_a(t)$ and $B_b(u) = g_b(t)$. If we can solve the (probably simpler) problems obtained by replacing one or the other of the functions g_a and g_b by zero, we can solve the original problem by adding the solutions to the two simpler problems.

Let us now turn to the inhomogeneous differential equation $L(u) = F(x,t)$. Suppose the homogeneous equation $L(u) = 0$ with homogeneous boundary conditions $B(u) = 0$ can be handled by separation of variables, leading to solutions $u(x,t) = \sum c_n\phi_n(x)\psi_n(t)$ where the ϕ_n's are the eigenfunctions for a Sturm-Liouville problem. Then the same sort of eigenfunction expansion can be used to produce solutions of the inhomogeneous equation $L(u) = F(x,t)$ subject to the same boundary conditions $B(u) = 0$. Namely, for each t we expand the function $F(x,t)$ in terms of the eigenfunctions $\phi_n(x)$,

$$F(x,t) = \sum c_n(t)\phi_n(x),$$

and we try to find a solution u in the form

$$u(x,t) = \sum \omega_n(t)\phi_n(x),$$

where the functions $\omega_n(t)$ are to be determined. If we plug these series into the differential equation $L(u) = F$, the result will be a sequence of *ordinary* differential equations for the unknown functions $\omega_n(t)$ in terms of the known functions $c_n(t)$. These equations can be solved subject to whatever initial conditions at $t = 0$ one may require. The resulting function $u(x,t)$ then satisfies the differential equation $L(u) = F$ and the desired initial conditions; it satisfies the boundary conditions $B(u) = 0$ because they are built into the eigenfunctions ϕ_n. In short, we have:

Technique 2: The Sturm-Liouville expansions used to solve $L(u) = 0$ *with homogeneous boundary conditions* $B(u) = 0$ *can also be used to solve the inhomogeneous equation* $L(u) = F(x,t)$ *with the same boundary conditions.*

Another useful device is available for solving (4.2) when the inhomogeneous terms F and g are independent of t:

$$L(u) = F(x), \qquad B(u) = c, \qquad I(u) = h(x). \tag{4.6}$$

(We have written c instead of g to remind ourselves that it is constant.) In this case, the differential equation $L(u) = F$ with boundary conditions $B(u) = c$ may admit **steady-state solutions**, that is, solutions that are independent of t. The superposition principle (Technique 1) can be used to break (4.6) down into the problem of finding a steady-state solution and solving the homogeneous equation with given initial conditions: If $u_0(x)$ and $v(x,t)$ satisfy

$$L(u_0) = F(x), \qquad B(u_0) = c, \tag{4.7}$$

$$L(v) = 0, \qquad B(v) = 0, \qquad I(v) = h(x) - u_0(x), \tag{4.8}$$

then $u(x,t) = u_0(x)+v(x,t)$ satisfies (4.6). (4.7) is relatively easy to solve because it is only an *ordinary* differential equation for u_0, and (4.8) is just (4.1) again (with different initial conditions). To summarize:

Technique 3: To solve an inhomogeneous problem with time-independent data, reduce to the homogeneous case by finding a steady-state solution.

Technique 3 is not infallible. Sometimes there is no steady-state solution; that is, the boundary conditions $B(u_0) = c$ are incompatible with the differential equation $L(u_0) = F(x)$ when u_0 is independent of t. (When this happens, there is usually a good physical reason for it.) We also observe that in the case of homogeneous boundary conditions $B(u) = 0$, Techniques 2 and 3 can both be used to solve the equation $L(u) = F(x)$. The solutions may differ in appearance (the first one involves a series expansion for F), but they are actually the same.

There remains the question of solving problems with inhomogeneous boundary conditions $B(u) = g(t)$ that are time-dependent. Often the most efficient tool for handling such problems is the Laplace transform; see §8.4. However, it is worth noting that the superposition principle can be used to trade off inhomogeneous boundary conditions for inhomogeneous equations. Namely, suppose we wish to solve

$$L(u) = 0, \qquad B(u) = g(t), \qquad I(u) = 0. \tag{4.9}$$

Let $w(x,t)$ be any smooth function that satisfies the boundary conditions $B(w) = g(t)$ and the initial conditions $I(w) = 0$; such functions are relatively easy to construct because no differential equation needs to be solved. But then u satisfies (4.9) if and only if $v = u - w$ satisfies

$$L(v) = F(x,t), \qquad B(v) = 0, \qquad I(v) = 0,$$

where $F(x,t) = -L(w)$. In this way, problem (4.4) can be reduced to problem (4.5), which we have already discussed.

The preceding discussion has been phrased in terms of time-dependent problems for ease of exposition, but the techniques we have presented apply equally well to problems not involving time, such as the Laplace equation in two space variables x and y, with one of these variables playing the role of t. Here there are no "initial conditions" as opposed to "boundary conditions" but rather boundary conditions pertaining to different parts of the boundary; and "steady-state solutions" are to be interpreted as solutions that depend on only one of the variables. But the same ideas still work.

4.2 One-dimensional heat flow

In §2.5 we solved the problem of finding the temperature $u(x,t)$ in a rod that is insulated along its length and occupies the interval $0 \leq x \leq l$, given that the ends of the rod are either (a) insulated or (b) held at temperature zero. (The reader may prefer to think instead of a slab occupying the region $0 \leq x \leq l$ of xyz-space, where conditions are such that variations in temperature in the yz-directions are insignificant. The mathematics is the same.) Here we play some more complicated variations on the same theme.

Newton's law of cooling

Consider the same rod as before, and suppose the ends of the rod are in contact with a medium at temperature zero; but now suppose that the boundary conditions are given by Newton's law of cooling: the temperature gradient across the ends is proportional to the temperature difference between the ends and the surrounding medium. That is, we have the boundary value problem

$$u_t = ku_{xx}, \qquad u_x(0,t) = \alpha u(0,t), \qquad u_x(l,t) = -\alpha u(l,t), \tag{4.10}$$

subject of course to an initial condition $u(x,0) = f(x)$. Here α is a positive constant; the fact that the coefficient is α at $x = 0$ and $-\alpha$ at $x = l$ expresses the fact that, if $u(x,t) > 0$, the temperature will be increasing as one crosses the boundary at $x = 0$ from left to right and decreasing as one crosses the boundary at $x = l$ from left to right (and vice versa if $u(x,t) < 0$). The cases of insulated boundary, or boundary held at temperature zero, are the limiting cases $\alpha \to 0$ and $\alpha \to \infty$ of this setup.

We apply separation of variables. As before, if we set $u(x,t) = X(x)T(t)$ in (4.10) and call the separation constant $-k\nu^2$, we obtain the differential equation $T' = -k\nu^2 T$ for T and the Sturm-Liouville problem

$$X'' + \nu^2 X = 0, \qquad X'(0) = \alpha X(0), \quad X'(l) = -\alpha X(l) \tag{4.11}$$

for X. We solved this problem in §3.5, and the analysis there shows the following:

(i) Zero is not an eigenvalue.

(ii) The positive eigenvalues are the numbers ν^2 such that ν satisfies

$$\tan \nu l = \frac{2\alpha\nu}{\nu^2 - \alpha^2},$$

and there is an infinite sequence $\nu_1 < \nu_2 < \cdots$ of such ν's. (See Figure 3.5 for the case $\alpha = 1$.) The normalized eigenfunction corresponding to the eigenvalue ν_n^2 is

$$\phi_n(x) = d_n^{-1}(\nu_n \cos \nu_n x + \alpha \sin \nu_n x)$$

where

$$d_n^2 = \tfrac{1}{2}(\nu_n^2 + \alpha^2)l + \alpha.$$

(iii) The negative eigenvalues are the numbers $-\mu^2$ such that μ satisfies

$$\tanh l\mu = \frac{-2\alpha\mu}{\mu^2 + \alpha^2},$$

but there are no solutions of this equation since the left and right sides always have opposite signs.

Now we can solve the boundary value problem (4.10) subject to the initial condition $u(x, 0) = f(x)$. Namely, we expand f in terms of the functions ϕ_n, which we know to be an orthonormal basis for $L^2(0, l)$: $f = \sum\langle f, \phi_n\rangle\phi_n$. Then we solve the differential equation $T' = -k\nu_n^2 T$ with initial value $\langle f, \phi_n\rangle$, obtaining $T(t) = \langle f, \phi_n\rangle \exp(-k\nu_n^2 t)$. Finally, we put it all together, obtaining

$$\begin{aligned} u(x,t) &= \sum_1^\infty \langle f, \phi_n\rangle \exp(-k\nu_n^2 t)\phi_n(x) \\ &= \sum_1^\infty \frac{2c_n}{(\nu_n^2 + \alpha^2)l + 2\alpha} \exp(-k\nu_n^2 t)(\nu_n \cos \nu_n x + \alpha \sin \nu_n x) \end{aligned}$$

where

$$c_n = \int_0^l f(x)(\nu_n \cos \nu_n x + \alpha \sin \nu_n x)\, dx.$$

Since all the eigenvalues are positive, the solution approaches zero exponentially fast as $t \to \infty$: this is just what one would expect physically.

Suppose we replace (4.10) by

$$u_t = ku_{xx}, \qquad u_x(0, t) = u(0, t), \quad u_x(l, t) = 4u(l, t). \tag{4.12}$$

Here the left boundary condition is just as before with $\alpha = 1$, but the right boundary condition is physically unreasonable: It says that heat is being pumped into the rod at the right end when the temperature of the rod is already greater than that of the surroundings, and sucked out when the temperature of the rod is less. Nonetheless, we can still solve the mathematical problem and see what we get. We solved the relevant Sturm-Liouville problem in §3.5, and we found that in addition to the sequence $\{\nu_n^2\}_1^\infty$ of positive eigenvalues, with eigenfunctions $\{\phi_n\}_1^\infty$, there is one negative eigenvalue $-\mu_0^2$, with eigenfunction ϕ_0. The solution of (4.12) with initial data f is then

$$u(x,t) = \langle f, \phi_0 \rangle \exp(k\mu_0^2 t)\phi_0(x) + \sum_1^\infty \langle f, \phi_n \rangle \exp(-k\nu_n^2 t)\phi_n(x).$$

Here the term involving the negative eigenvalue *grows* exponentially as $t \to \infty$, unless by some chance $\langle f, \phi_0 \rangle = 0$. But this is to be expected: If the rod is initially hot, it keeps getting hotter because heat is being pumped in at the right end. So the mathematics still makes some physical sense even when the physics is unrealistic!

Inhomogeneous boundary conditions

So far we have always assumed that both ends of the rod or slab are exposed to the same outside temperature. But perhaps the rod goes through a wall (or the slab *is* a wall) between two rooms at different temperatures: The temperature on the left is zero, for instance, and the temperature on the right is $A \neq 0$. Then we should impose the boundary conditions

$$u(0,t) = 0, \qquad u(l,t) = A \tag{4.13}$$

or, for Newton's law of cooling,

$$u_x(0,t) = \alpha u(0,t), \qquad u_x(l,t) = -\alpha[u(l,t) - A]. \tag{4.14}$$

These are inhomogeneous boundary conditions that do not depend on time, so we apply Technique 3 of §4.1 to find a solution. That is, we begin by finding the steady-state solution $u_0(x)$ of the heat equation that satisfies (4.13) or (4.14). This is easy: For a function u_0 that does not depend on t, the heat equation simply becomes $u_0'' = 0$. The general solution of this equation is $u_0(x) = cx + d$, and we have merely to determine the coefficients c and d so that u_0 satisfies (4.13) or (4.14). For (4.13) the solution is

$$u_0(x) = (A/l)x,$$

and for (4.14) the solution is

$$u_0(x) = \frac{A}{2 + \alpha l}(\alpha x + 1).$$

Now we can solve the heat equation with initial data $u(x,0) = f(x)$, subject to the boundary conditions (4.13) or (4.14) — let us say (4.13), to be definite. Namely, we set

$$u(x,t) = u_0(x) + v(x,t) = (A/l)x + v(x,t).$$

Then u satisfies

$$u_t = ku_{xx}, \quad u(0,t) = 0, \quad u(l,t) = A, \quad u(x,0) = f(x)$$

if and only if v satisfies

$$v_t = kv_{xx}, \quad v(0,t) = v(l,t) = 0, \quad v(x,0) = f(x) - (A/l)x.$$

Thus we now have *homogeneous* boundary conditions for v, with slightly different initial data. As we know from §2.5, we can solve this problem by expanding $f(x) - (A/l)x$ in a Fourier sine series; and we have essentially computed the Fourier sine series for $(A/l)x$ in §2.1:

$$\frac{A}{l}x = \sum_1^\infty \frac{2A(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{l}.$$

The result is

$$v(x,t) = \sum_1^\infty \left(b_n - \frac{2A(-1)^{n+1}}{n\pi} \right) e^{-n^2\pi^2 kt/l^2} \sin\frac{n\pi x}{l},$$
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx,$$

and hence

$$\begin{aligned} u(x,t) &= \frac{A}{l}x + v(x,t) \\ &= \sum_1^\infty \frac{2A(-1)^{n+1}}{n\pi}(1 - e^{-n^2\pi^2 kt/l^2}) \sin\frac{n\pi x}{l} + \sum_1^\infty b_n e^{-n^2\pi^2 kt/l^2} \sin\frac{n\pi x}{l}. \end{aligned}$$

The first sum here represents the solution that starts at 0 at time $t = 0$ and rises to the steady state $(A/l)x$ because of the influx of heat from the right, whereas the second sum represents the transient effects of the initial temperature $f(x)$.

The inhomogeneous heat equation

Having considered inhomogeneous boundary conditions, we now consider the inhomogeneous differential equation $u_t = ku_{xx} + F(x,t)$. Here $F(x,t)$ models the effect of some mechanism that adds or subtracts heat from the rod — perhaps some heat sources along the length of the rod, or a chemical or nuclear reaction within the rod itself. (F is measured in degrees per unit time; it represents the rate at which heat is being produced.) To be definite, let us suppose that the rod is initially at temperature zero and is held at temperature zero at both ends; thus, we wish to solve

$$u_t = ku_{xx} + F(x,t), \quad u(x,0) = 0, \quad u(0,t) = u(l,t) = 0. \tag{4.15}$$

If the inhomogeneous term is independent of t, i.e., $F(x,t) = F(x)$, we can use the same device as in the previous example. That is, we first solve the steady-state problem

$$ku_0'' + F(x) = 0, \qquad u_0(0) = u_0(l) = 0,$$

which is easily accomplished by integrating $-F(x)/k$ twice and choosing the constants of integration appropriately. Then the substitution $u(x,t) = u_0(x) + v(x,t)$ turns (4.15) into

$$v_t = kv_{xx}, \qquad v(x,0) = -u_0(x), \qquad v(0,t) = v(l,t) = 0,$$

which we have already solved by means of Fourier sine series.

For the general case, we can use Technique 2 of §4.1. The eigenfunction expansion that solves the homogeneous case $F = 0$ is the Fourier sine series. Hence, we begin by expanding everything in sight in a Fourier sine series:

$$u(x,t) = \sum_1^\infty b_n(t) \sin \frac{n\pi x}{l}, \qquad F(x,t) = \sum_1^\infty \beta_n(t) \sin \frac{n\pi x}{l}. \tag{4.16}$$

Here the coefficients β_n are computed from the known function F in the usual way, and the coefficients b_n are to be determined. If we plug these series into the differential equation (4.15), we obtain

$$\sum_1^\infty b_n'(t) \sin \frac{n\pi x}{l} = \sum_1^\infty \left(-\frac{n^2\pi^2 k}{l^2} b_n(t) + \beta_n(t) \right) \sin \frac{n\pi x}{l},$$

and equating the coefficients of $\sin(n\pi x/l)$ on both sides yields

$$b_n'(t) + \frac{n^2\pi^2 k}{l^2} b_n(t) = \beta_n(t).$$

This is a first-order ordinary differential equation for b_n, and it is easily solved by multiplying through by the integrating factor $e^{n^2\pi^2 kt/l^2}$:

$$\frac{d}{dt}\left[b_n(t) \exp\left(\frac{n^2\pi^2 kt}{l^2}\right)\right] = \beta_n(t) \exp\left(\frac{n^2\pi^2 kt}{l^2}\right).$$

Integrating both sides and remembering that $b_n(0) = 0$ (from the initial condition in (4.15)), we find that

$$b_n(t) = \exp\left(-\frac{n^2\pi^2 kt}{l^2}\right) \int_0^t \beta_n(s) \exp\left(\frac{n^2\pi^2 ks}{l^2}\right) ds, \tag{4.17}$$

and the solution u is obtained by substituting this formula for b_n into (4.16).

The sharp-eyed reader will have noticed that this line of argument needs some justification. We differentiated the series $u = \sum b_n(t)\sin(n\pi x/l)$ termwise with respect to t and x without really knowing what we were doing, since the coefficients b_n (and hence the convergence properties of the series) were as yet unknown. Only after we have found formula (4.17) and substituted it into (4.16) can we see that the function u thus defined really solves the problem. (It always does so in the weak sense discussed in §9.5. Moreover, if the function $F(x,t)$ is such that its Fourier sine coefficients β_n tend to zero reasonably rapidly as $n \to \infty$, the same will be true of the coefficients b_n of u in view of (4.17), and one can then show that u satisfies (4.15) in the ordinary pointwise sense.)

Example. Suppose the rod is radioactive and produces heat at a constant rate R. Thus the problem to be solved is

$$u_t = ku_{xx} + R, \qquad u(x,0) = 0, \qquad u(0,t) = u(l,t) = 0.$$

Employing Technique 3, we first solve

$$u_0''(x) = -R/k, \qquad u_0(0) = u_0(l) = 0,$$

to obtain

$$u_0(x) = (R/2k)x(l-x).$$

Next we solve

$$v_t = kv_{xx}, \qquad v(x,0) = -u_0(x), \qquad v(0,t) = v(l,t) = 0$$

by expanding u_0 in its Fourier sine series (cf. Exercise 10, §2.4)

$$\frac{R}{2k}x(l-x) = \frac{4l^2R}{\pi^3 k}\sum_{n=1,3,5,\ldots}\frac{1}{n^3}\sin\frac{n\pi x}{l} \qquad (0 < x < l)$$

to obtain

$$v(x,t) = -\frac{4l^2R}{\pi^3 k}\sum_{n=1,3,5,\ldots}\frac{1}{n^3}\exp\frac{-n^2\pi^2 kt}{l^2}\sin\frac{n\pi x}{l}$$

and hence

$$\begin{aligned} u(x,t) &= u_0(x) + v(x,t) \\ &= \frac{4l^2R}{\pi^3 k}\sum_{n=1,3,5,\ldots}\frac{1}{n^3}\left(1 - \exp\frac{-n^2\pi^2 kt}{l^2}\right)\sin\frac{n\pi x}{l}. \end{aligned} \tag{4.18}$$

(See Figure 4.1.)

Employing Technique 2, we expand $u(x,t)$ and the constant function R in Fourier sine series (cf. Exercise 1, §2.4):

$$u(x,t) = \sum_1^\infty b_n(t)\sin\frac{n\pi x}{l}, \qquad R = \frac{4R}{\pi}\sum_{n=1,3,5,\ldots}\frac{1}{n}\sin\frac{n\pi x}{l} \qquad (0 < x < l).$$

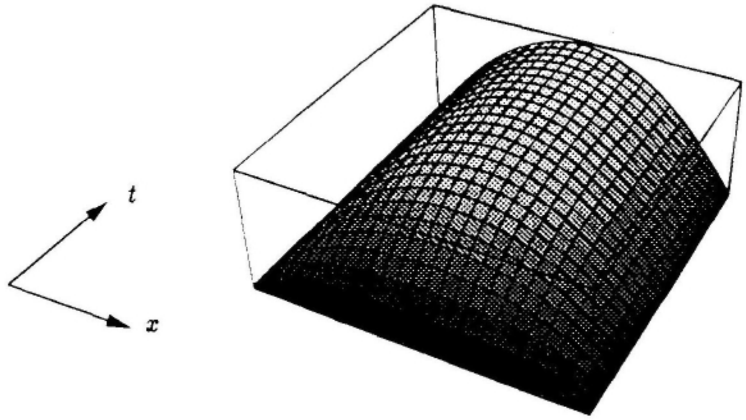


FIGURE 4.1. The temperature function $u(x,t)$ given by (4.18) with $k = 0.5$, $R = 1.3$, and $l = 1$, on the region $0 \le x \le 1$, $0 \le t \le 1$.

The differential equation for u then gives

$$b_n'(t) + \frac{n^2\pi^2 k}{l^2} b_n(t) = \begin{cases} 4R/n\pi, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

The solution to this equation with initial value 0 is

$$b_n(t) = \frac{4l^2 R}{n^3\pi^3 k}\left(1 - \exp\frac{-n^2\pi^2 kt}{l^2}\right)$$

for n odd and $b_n(t) = 0$ for n even, which again gives the solution (4.18).

Heat flow in nonuniform materials

One can also consider heat flow in rods or slabs of nonuniform composition, where the specific heat density σ and the thermal conductivity K vary from point to point. In this case the (homogeneous) heat equation becomes $\sigma(x)u_t = (K(x)u_x)_x$ (see Appendix 1). All of what we have done works in principle for this more general situation; the difference is that one must solve boundary value problems for the Sturm-Liouville equation $(K(x)f')' + \lambda\sigma(x)f = 0$ rather than $kf'' + \lambda f = 0$ (with k constant).

EXERCISES

All these problems concern heat flow in a rod on the interval $[0, l]$; in all except the last one, it is assumed that heat can enter or leave the rod only at the ends.

1. Suppose the end $x = 0$ is held at temperature zero while the end $x = l$ is insulated.
   a. Find a series expansion for the temperature $u(x,t)$ given the initial temperature $f(x)$.
   b. What is $u(x,t)$ when $f(x) \equiv 50$?

2. Repeat Exercise 1a, replacing the assumption $u(0,t) = 0$ by the assumption $u(0,t) = C \neq 0$.
3. Repeat Exercise 1a, replacing the assumption that $u_x(l,t) = 0$ by the assumption $u_x(l,t) = A$ (i.e., heat is being supplied at a constant rate at the right end).
4. Repeat Exercise 1a, assuming that the rod generates heat within itself at a constant rate R, so the heat equation is replaced by $u_t = ku_{xx} + R$.
5. Take $l = \pi$ and solve: $u_t = ku_{xx} + e^{-2t}\sin x$, $u(x,0) = u(0,t) = u(\pi,t) = 0$.
6. In the example of the radioactive rod, suppose that the reaction that produces the heat inside the rod dies out over time, so that the differential equation is $u_t = ku_{xx} + Re^{-ct}$. What is the solution?
7. Suppose that a rod is insulated at both ends, has initial temperature zero, and generates heat within itself at the constant rate R; thus, $u_t = ku_{xx} + R$ and $u(x,0) = u_x(0,t) = u_x(l,t) = 0$.
   a. Show that Technique 3 doesn't work here; that is, there is no steady-state solution of $u_t = ku_{xx} + R$, $u_x(0,t) = u_x(l,t) = 0$. Why is this to be expected physically?
   b. Solve the problem by Technique 2 (or by making a clever guess).
   c. Solve the problem with the constant R replaced by Re^{-ct}.
8. Solve: $u_t = ku_{xx}$, $u_x(0,t) = 0$, $u_x(l,t) + bu(l,t) = 0$ $(b > 0)$, $u(x,0) = 100$. (Cf. Exercise 5, §3.5. What is the physical interpretation?)
9. Let $k(x)$ be a smooth positive function on $[0,l]$. Solve the boundary value problem $u_t = (ku_x)_x + f(x,t)$, $u(0,t) = u(l,t) = u(x,0) = 0$, in terms of the eigenvalues $\{\lambda_n\}$ and the eigenfunctions $\{\phi_n\}$ of the Sturm-Liouville problem $(kf')' + \lambda f = 0$, $f(0) = f(l) = 0$.
10. We have always supposed that the rod is insulated along its length. Suppose instead that the surroundings are at temperature zero, and that heat transfer takes place at a rate proportional to the temperature difference (Newton's law). A reasonable model for this situation is the modified heat equation $u_t = ku_{xx} - hu$, where h is a positive constant.
    a. Show that u satisfies this equation if and only if $u(x,t) = e^{-ht}v(x,t)$ where v satisfies the ordinary heat equation. Show also how this result could be discovered by separation of variables.
    b. Suppose that both ends are insulated and that the initial temperature is $f(x) = x$. Solve for $u(x,t)$.
    c. Suppose instead that one end is held at temperature 0 and the other is held at temperature 100, and that the initial temperature is zero. Solve for $u(x,t)$. (Use Technique 3, and cf. entry 20 in Table 1, §2.1.)

4.3 One-dimensional wave motion

In §2.5 we analyzed the vibrating string problem,

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(l,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

by means of Fourier sine series. We now consider some related boundary value problems.

The inhomogeneous wave equation

We can add an inhomogeneous term to the vibrating string problem to represent an external force that affects the vibrations:

$$\begin{aligned} &u_{tt} = c^2 u_{xx} + F(x,t), \\ &u(0,t) = u(l,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x). \end{aligned} \tag{4.19}$$

For example, if the string is an electrically charged wire, F could result from a surrounding electromagnetic field.

The techniques that we developed in §4.1 and used in §4.2 to solve the inhomogeneous heat equation work equally well here. If F is independent of t, one can first find a steady-state solution $u_0(x)$ by integrating F twice and then solve the homogeneous wave equation for $v = u - u_0$ with the initial displacement f replaced by $f - u_0$. Or, for the general case, one can expand $u(x,t)$ and $F(x,t)$ in Fourier sine series in x for each t,

$$u(x,t) = \sum_1^\infty b_n(t) \sin\frac{n\pi x}{l}, \qquad F(x,t) = \sum_1^\infty \beta_n(t) \sin\frac{n\pi x}{l},$$

yielding a sequence of ordinary differential equations for the Fourier coefficients of u in terms of those of F, namely,

$$b_n''(t) + \frac{n^2\pi^2c^2}{l^2} b_n(t) = \beta_n(t), \tag{4.20}$$

These equations can be solved by standard techniques such as variation of parameters or Laplace transforms (see §8.3 or Boyce-diPrima [10]); the solution with initial conditions $b_n(0) = b_n'(0) = 0$ is

$$b_n(t) = \frac{l}{n\pi c}\int_0^t \sin\frac{n\pi c(t-s)}{l}\,\beta_n(s)\,ds. \tag{4.21}$$

This formula leads to the solution of (4.19) with initial conditions $f = g = 0$. But then to solve (4.19) with arbitrary initial data f and g, by the superposition principle (Technique 1) one is reduced to solving the *homogeneous* wave equation with these initial data; and this we have already done. (As with the heat equation, these calculations show only that $u = \sum b_n(t)\sin(n\pi x/l)$, with $b_n(t)$ defined by (4.21), is a reasonable candidate for a solution; further arguments are needed for a rigorous establishment of the fact that it really works.)

Vibrations with free ends

Another boundary value problem of interest for the wave equation is obtained by requiring that the spatial derivative u_x rather than u itself should vanish at the endpoints:

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \\ &u_x(0,t) = u_x(l,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x). \end{aligned} \tag{4.22}$$

If one thinks of a vibrating string, this represents a string whose ends are free to move on frictionless tracks along the lines $x = 0$ and $x = l$ in the xu-plane. The condition that $u_x = 0$ at the endpoints expresses the fact that there are no forces directed along the tracks to oppose the tension in the string. This may seem a rather artifical situation, but more natural interpretations of (4.22) are available. For one, (4.22) is a model for the longitudinal vibrations of an elastic rod or a spring that is free at both ends. ("Longitudinal" means that the vibrations involve displacements of the material along the x-axis by compression and extension of the rod or spring, rather than displacements perpendicular to the x-axis as in the vibrating string.) An even better interpretation of (4.22) is as the longitudinal vibrations of a column of air that is open at both ends, such as a flute or organ pipe. In the case of the flute, for example, musical notes are produced by vibrations of the air within the flute; these vibrations are largely confined to the interval between the hole where the moving air is introduced by the player and the first open finger-hole.

The mathematics of (4.22) is very similar to the vibrating string problem, except that one uses Fourier cosine series rather than sine series. Indeed, we can solve the problem by expanding everything in Fourier cosine series from the outset: If we substitute

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos\frac{n\pi x}{l}, \qquad g(x) = \tfrac{1}{2}\alpha_0 + \sum_1^\infty \alpha_n \cos\frac{n\pi x}{l},$$
$$u(x,t) = \tfrac{1}{2}A_0(t) + \sum_1^\infty A_n(t) \cos\frac{n\pi x}{l},$$

into (4.22), we obtain the ordinary differential equations

$$A_n''(t) = -\frac{n^2\pi^2c^2}{l^2}A_n(t), \qquad A_n(0) = a_n, \quad A_n'(0) = \alpha_n.$$

The solution to this, for $n > 0$, is

$$A_n(t) = a_n \cos\frac{n\pi ct}{l} + \frac{l\alpha_n}{n\pi c}\sin\frac{n\pi ct}{l},$$

whereas for $n = 0$ it is

$$A_0(t) = a_0 + \alpha_0 t.$$

Hence, the solution u of (4.22) is given by

$$u(x,t) = \tfrac{1}{2}(a_0 + \alpha_0 t) + \sum_1^\infty \cos\frac{n\pi x}{l}\left(a_n \cos\frac{n\pi ct}{l} + \frac{l\alpha_n}{n\pi c}\sin\frac{n\pi ct}{l}\right). \qquad (4.23)$$

(As usual, the formal differentiations used in arriving at this formula need to be justified after the fact. Alternatively, one could arrive at (4.23) by separation of variables.)

Here there is a bit of a surprise. The terms with $n > 0$ describe the vibratory motion of the string (or rod, or spring, or whatever; let us call it the "device"), and the term $\frac{1}{2}a_0$ is just a constant displacement, of no importance; but if $\alpha_0 \neq 0$, the term $\frac{1}{2}\alpha_0 t$ says that the device as a whole is moving with velocity $\frac{1}{2}\alpha_0$ — perpendicular to the x-axis in the case of a string, and along the x-axis in the other cases. Indeed, there is nothing in the setups we have described to prevent this, since the ends of the device are free to move. The constant $\frac{1}{2}\alpha_0 = l^{-1}\int_0^l g(x)\,dx$ is the average initial velocity of the device; and in the absence of any countervailing forces the device will continue to move with this velocity. If the device as a whole stays put, it simply means that $\alpha_0 = 0$.

Mixed boundary conditions

Another problem of interest is the one with mixed boundary conditions:

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \\ &u(0,t) = u_x(l,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x). \end{aligned} \qquad (4.24)$$

Here the left endpoint is fixed and the right endpoint is free. One can think of a string or elastic rod with one fixed end and one free end, or a column of air that is closed at one end and open at the other, such as a clarinet or a stopped organ pipe.

After separation of variables, the Sturm-Liouville problem to be solved in this case is

$$X'' + \lambda X = 0, \qquad X(0) = X'(l) = 0.$$

This was the subject of Exercise 3, §3.5, but we shall briefly derive the solution here. If we set $\lambda = \nu^2$, the general solution of the differential equation is a linear combination of $\sin\nu x$ and $\cos\nu x$. The condition $X(0) = 0$ implies that $X(x) = c\sin\nu x$, and the condition $X'(l) = 0$ then becomes $\cos\nu l = 0$. This means that νl must be a half-integer multiple of π, so the (unnormalized) eigenfunctions are

$$X_n(x) = \sin\frac{(n-\frac{1}{2})\pi x}{l} = \sin\frac{(2n-1)\pi x}{2l}, \qquad n = 1, 2, 3, \ldots$$

We leave to the reader to work out the details, but it should be pretty clear now that the solution $u(x,t)$ of (4.22) will have the form

$$\sum_1^\infty \sin\frac{(2n-1)\pi x}{2l}\left[a_n\cos\frac{(2n-1)\pi ct}{2l} + \frac{2l\alpha_n}{(2n-1)\pi c}\sin\frac{(2n-1)\pi ct}{2l}\right]. \qquad (4.25)$$

There is an interesting difference between the frequency spectra of the waves (4.23) and (4.25). The nth term of the series in (4.23) represents a vibration with period $2l/nc$, or frequency $nc/2l$; so the allowable frequencies are the integer multiples of the "fundamental" frequency $c/2l$. The nth term in (4.25), on the other hand, has frequency $(2n-1)c/4l$; so the allowable frequencies are the *odd* integer multiples of the fundamental frequency $c/4l$. In particular, the fundamental frequency in the former case is twice as great as in the latter. In musical terms, this means that an air column open at only one end produces notes an octave lower than a column of the same length open at both ends, and that its odd harmonics are missing. (See Figure 4.2.)

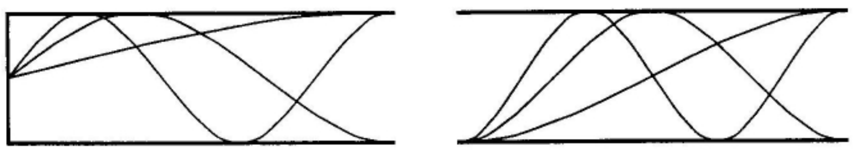


FIGURE 4.2. Profiles of the vibrations corresponding to the three lowest eigenvalues in a pipe open at one end (left) and a pipe open at both ends (right).

These remarks apply to clarinets but not to oboes, saxophones, or any of the brass instruments. Oboes and saxophones have conical bores (their interior diameter increases steadily from mouthpiece to bell) rather than the cylindrical bore of the clarinet (whose interior diameter is essentially constant). The effect of this, as we shall see in §5.6, is that the frequencies they produce are about the same as the frequencies of a cylindrical column of the same length that is *open at both ends*. In particular, they produce all integer multiples of the fundamental frequency. The physics of the brass instruments is considerably more complex, and we shall not discuss it here. For further information on the physics of musical instruments, we refer the reader to Hutchins [31] and Taylor [51].

Other problems in wave motion

A number of other variations on these themes are possible. For example, one can add an inhomogeneous term to the wave equation in (4.22) and (4.24), just as in (4.19), and the same techniques of solution are applicable. One can also consider inhomogeneous boundary conditions, such as

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = 0, \quad u(l,t) = h(t).$$

This might represent a string that is fixed at the left end and is being shaken at the right end, or an electromagnetic signal being sent down a wire from the end $x = l$. We shall solve this problem in §8.4 by using the Laplace transform; for the time being, we leave it to the reader to work out a special case in Exercise 7, using the trick mentioned at the end of §4.1.

One can also consider waves in nonuniform media — for example, a vibrating string whose linear mass density ρ varies from point to point. (Perhaps the string is thicker in some places than in others.) In this case the wave equation becomes

$$u_{tt} = T\rho(x)^{-1}u_{xx} \tag{4.26}$$

where T is the tension of the string (see Appendix 1). The Sturm-Liouville equation that results from separation of variables is then $f'' + \lambda\rho(x)f = 0$, which produces eigenfunctions that are orthogonal with respect to the weight $\rho(x)$. One can then solve the wave equation (4.26) by using these eigenfunctions in place of sines and cosines.

EXERCISES

1. Verify that the function $b_n(t)$ defined by (4.21) satisfies the differential equation (4.20) and the initial conditions $b_n(0) = b_n'(0) = 0$.
2. One end of an elastic bar of length l is held at $x = 0$, and the other end is stretched from its natural position $x = l$ to $x = (1 + b)l$. Thus, an arbitrary point x in the bar is moved to $(1+b)x$, so its displacement from equilibrium is bx. At time $t = 0$ the ends of the bar are released; thus, $u(x,0) = bx$ and $u_t(x,0) = 0$.
   a. Find the displacement $u(x,t)$ at times $t > 0$.
   b. Show that the velocity at the left end of the bar alternately takes the values bc and $-bc$ on time intervals of length l/c. (That is, $u_t(0,t) = bc$ for $2ml/c < t < (2m+1)l/c$ and $u_t(0,t) = -bc$ for $(2m+1)l/c < t < (2m+2)l/c$, $m = 0, 1, 2, \ldots$. Hint: Entry 6 of Table 1, §2.1.)
3. Suppose a horizontally stretched string is heavy enough for the effects of gravity to be significant, so that the wave equation must be replaced by $u_{tt} = c^2u_{xx} - g$ where g is the acceleration of gravity. (The boundary conditions are still $u(0,t) = u(l,t) = 0$.)
   a. Find the steady-state solution $\phi(x)$.
   b. Suppose that initially $u(x,0) = u_t(x,0) = 0$. Find the solution $u(x,t)$ as a Fourier series, and show that

   $$u(x,t) = \phi(x) - \tfrac{1}{2}\left[\Phi(x+ct) + \Phi(x-ct)\right]$$

   where Φ is the odd $2l$-periodic extension of ϕ. (Cf. the discussion in §2.5.)
4. In problem (4.22) discussed in the text, assume that $\int_0^l g(x)\,dx = 0$ (average initial velocity is zero), and let $h(x) = \int_0^x g(\xi)\,d\xi$. Show that the solution (4.23) can be written as

   $$u(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\left[H(x+ct) - H(x-ct)\right]$$

   where F and H are the even $2l$-periodic extensions of f and h. (Cf. the discussion in §2.5.)

5. Find the general solution of $u_{tt} = c^2u_{xx} - a^2u$, $u(0,t) = u(l,t) = 0$, with arbitrary initial conditions. This is a model for a string vibrating in an elastic medium; the term $-a^2u$ represents the force of reaction of the medium on the string. (Hint: The differential equation is homogeneous; start from scratch with separation of variables.)
6. In real-life vibrating strings, the vibrations damp out because the strings are not perfectly elastic. This situation can be modeled by the modified wave equation $u_{tt} = c^2u_{xx} - 2ku_t$; the term $-2ku_t$ represents the frictional forces that cause the damping. (The factor of 2 is purely for convenience.) Find the general solution, subject to the boundary conditions $u(0,t) = u(l,t) = 0$. Assume at first that $k < \pi c/l$. What happens if $k \geq \pi c/l$? (See the hint for Exercise 5.)
7. A string of length $l = \pi$ (for simplicity) is fixed at one end and attached to an oscillator at the other, so that $u(0,t) = 0$ and $u(\pi,t) = \sin kt$. If the string is initially at rest ($u(x,0) = u_t(x,0) = 0$), find $u(x,t)$. (Hints: (1) Let $u(x,t) = v(x,t) + (x/\pi)\sin kt$ and solve for v. (2) When $k \neq \alpha$ the general solution of $f'' + \alpha^2 f = \beta \sin kt$ is $c_1 \cos \alpha t + c_2 \sin \alpha t + (\beta \sin kt)/(\alpha^2 - k^2)$.) The typical case is when k/c is not an integer; if it is, the answer will have a different form due to resonance between the imposed oscillations and one of the natural frequencies of the string.
8. The total energy of a vibrating string at time t, up to a constant factor, is

$$E(t) = \int_0^l \left[u_t(x,t)^2 + c^2u_x(x,t)^2\right] dx.$$

(The first term is the kinetic energy and the second term is the potential energy. u is assumed to be real here.)

   a. If the string has fixed ends and $u(x,t)$ is written as a Fourier series as in equation (2.24), show that

$$E(t) = \frac{\pi^2c^2}{2l}\sum_1^\infty (nb_n)^2 + \frac{l}{2}\sum_1^\infty B_n^2.$$

   (In particular, we have conservation of energy: $E(t)$ is independent of t. This also suggests that a natural physical requirement is that the series $\sum(nb_n)^2$ and $\sum B_n^2$ be convergent. This is the case if $u(x,0)$ is continuous and piecewise smooth and $u_t(x,0)$ is piecewise continuous. Why?)

   b. Derive a similar result for a vibrating string (or bar or air column) with free ends, with the same formula for $E(t)$.

4.4 The Dirichlet problem

The **Dirichlet problem** is to find a solution of Laplace's equation in a region D that assumes given values on the boundary ∂D of D:

$$\nabla^2 u = 0 \text{ in } D, \qquad u(x) = f(x) \quad \text{for } x \in \partial D.$$

This can be interpreted physically as finding the steady-state temperature in D when the temperature on ∂D is known, or as finding the electrostatic potential in the charge-free region D when the potential on ∂D is known. This problem can be studied in any number of dimensions; here we consider the 2-dimensional case for certain simple regions in which the method of separation of variables is effective. Some other boundary value problems for the equation $\nabla^2 u = 0$ are considered in the exercises.

The Dirichlet problem in a rectangle

The simplest situation is that of a rectangle. We take the sides of the rectangle to have length l and L, and we take the origin to be at the lower left corner. Thus,

$$D = [0,l] \times [0,L] = \left\{(x,y) : 0 \le x \le l,\ 0 \le y \le L\right\},$$

and the boundary value problem to be solved is

$$u_{xx} + u_{yy} = 0,$$
$$u(x,0) = f_1(x), \quad u(x,L) = f_2(x), \quad u(0,y) = g_1(y), \quad u(l,y) = g_2(y).$$

By the superposition principle (Technique 1) it will suffice to solve this problem in the special cases $g_1 = g_2 = 0$ and $f_1 = f_2 = 0$, as the solution in the general case is obtained by adding together the solutions for these two special cases. Moreover, the cases $g_1 = g_2 = 0$ and $f_1 = f_2 = 0$ are equivalent, just by interchanging the roles of x and y, so we work out only the first one:

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \\ &u(0,y) = u(l,y) = 0, \quad u(x,0) = f_1(x), \quad u(x,L) = f_2(x). \end{aligned} \tag{4.27}$$

We apply separation of variables. Neglecting the inhomogeneous boundary conditions for the moment, we search for solutions u that satisfy the homogeneous boundary conditions. Taking $u(x,y) = X(x)Y(y)$, we find from the differential equation that $X''Y + Y''X = 0$, or $Y''/Y = -X''/X$. Setting Y''/Y and $-X''/X$ equal to a constant ν^2, we obtain

$$X'' + \nu^2 X = 0, \quad X(0) = X(l) = 0,$$
$$Y'' - \nu^2 Y = 0.$$

The Sturm-Liouville problem for X is a familiar one that we have seen many times before: The eigenvalues are $\nu^2 = (n\pi/l)^2$ where n is a positive integer, and the corresponding eigenfunctions are $\sin(n\pi x/l)$. In other words, we are working once again with Fourier sine series in x. (Readers who foresaw this outcome immediately upon looking at (4.27) are to be congratulated on their instincts.) As for Y, the general solution of the equation $Y'' - \nu^2 Y = 0$ with $\nu^2 = (n\pi/l)^2$

is a linear combination of $\cosh(n\pi y/l)$ and $\sinh(n\pi y/l)$, so we are looking at solutions u of the form

$$u(x,y) = \sum_1^\infty \sin\frac{n\pi x}{l}\left(\alpha_n \cosh\frac{n\pi y}{l} + \beta_n \sinh\frac{n\pi y}{l}\right), \tag{4.28}$$

and we must determine the coefficients α_n and β_n to get the right boundary conditions at $y = 0$ and $y = L$ ("initial" and "final" conditions, if you like). We expand the functions f_1 and f_2 in (4.27) in their Fourier sine series:

$$f_1(x) = \sum_1^\infty a_n \sin\frac{n\pi x}{l}, \qquad f_2(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l}.$$

On setting $y = 0$ or $y = L$ in (4.28) and comparing coefficients, we find that

$$\alpha_n = a_n, \qquad \alpha_n \cosh\frac{n\pi L}{l} + \beta_n \sinh\frac{n\pi L}{l} = b_n,$$

or

$$\alpha_n = a_n, \qquad \beta_n = b_n \operatorname{csch}\frac{n\pi L}{l} - a_n \coth\frac{n\pi L}{l}.$$

The solution is obtained by substituting these formulas into (4.28). It can be expressed more symmetrically by taking $\sinh[n\pi(L-y)/l]$ and $\sinh(n\pi y/l)$ as a basis for solutions to $Y'' - (n\pi/l)^2 Y = 0$ instead of $\cosh(n\pi y/l)$ and $\sinh(n\pi y/l)$; the result is

$$u(x,y) = \sum_1^\infty \sin\frac{n\pi x}{l}\left(A_n \sinh\frac{n\pi(L-y)}{l} + B_n \sinh\frac{n\pi y}{l}\right),$$

$$A_n = a_n \operatorname{csch}\frac{n\pi L}{l}, \qquad B_n = b_n \operatorname{csch}\frac{n\pi L}{l}.$$

The Dirichlet problem in polar coordinates

We next solve the Dirichlet problem in a "polar-coordinate rectangle"

$$S = \left\{(r\cos\theta, r\sin\theta) : r_0 \le r \le r_1,\ \alpha \le \theta \le \beta\right\}.$$

(See Figure 4.3.) For this we need the formula for the Laplacian in polar coordinates:

$$\nabla^2 u = u_{xx} + u_{yy} = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}.$$

This formula is derived in Appendix 4.

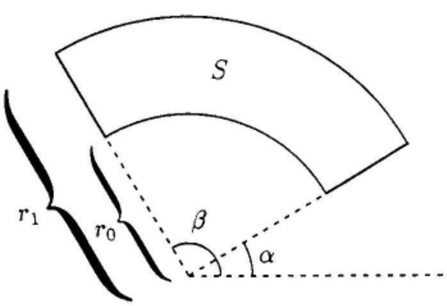


FIGURE 4.3. A "rectangular" region in polar coordinates.

In order to solve the Dirichlet problem on the region S, as in the rectangular case it will suffice to do the special cases when the solution is to vanish on the two radial pieces of the boundary or on the two circular pieces of the boundary. We shall work out the first of these cases here and leave the second one as Exercise 7. By rotating the coordinates suitably we may assume that the initial angle α is 0, so the problem we are to solve is

$$\begin{aligned} &u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0 \quad \text{in } S, \\ &u(r,0) = u(r,\beta) = 0, \quad u(r_1,\theta) = f(\theta), \quad u(r_0,\theta) = g(\theta). \end{aligned} \tag{4.29}$$

As usual, we begin by looking for product solutions $u(r,\theta) = R(r)\Theta(\theta)$ that satisfy the homogeneous boundary conditions. For such a u, Laplace's equation becomes

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)},$$

so upon setting both these expressions equal to a constant ν^2 we obtain

$$\Theta''(\theta) + \nu^2\Theta(\theta) = 0, \qquad \Theta(0) = \Theta(\beta) = 0, \tag{4.30}$$

$$r^2R''(r) + rR'(r) - \nu^2R(r) = 0. \tag{4.31}$$

The Sturm-Liouville problem (4.30) is our old friend that leads to the eigenvalues $\nu^2 = (n\pi/\beta)^2$ and eigenfunctions $\sin(n\pi\theta/\beta)$. The equation (4.31) for R is a special case of the **Euler equation**

$$r^2f''(r) + arf'(r) + bf(r) = 0, \tag{4.32}$$

which is one of the few types of equations with variable coefficients that can be solved in an elementary way. Namely, just as one uses exponential functions to solve constant-coefficient equations, one uses power functions to solve the Euler equation. Substituting $f(r) = r^\lambda$ in (4.32) yields

$$\left[\lambda(\lambda-1) + a\lambda + b\right]r^\lambda = 0,$$

so if λ_1 and λ_2 are the roots of the quadratic polynomial $\lambda^2 + (a-1)\lambda + b$, the functions r^{λ_1} and r^{λ_2} satisfy (4.32). The general solution of (4.32) is then a linear combination of these two except when $\lambda_1 = \lambda_2$, in which case the general solution is a linear combination of r^{λ_1} and $r^{\lambda_1}\log r$.

In the case (4.31) with which we are concerned, we have $a = 1$ and $b = \nu^2 = (n\pi/\beta)^2$, so the quadratic polynomial becomes $\lambda^2 - (n\pi/\beta)^2$, whose roots are $\lambda = \pm n\pi/\beta$. Therefore, we have found the following sort of solutions to Laplace's equation in the region S:

$$u(r,\theta) = \sum_1^\infty \sin\frac{n\pi\theta}{\beta}\left(a_nr^{n\pi/\beta} + b_nr^{-n\pi/\beta}\right).$$

It remains only to choose a_n and b_n to satisfy the remaining boundary conditions in (4.29). But this is easy: If we expand f and g in their Fourier sine series,

$$f(\theta) = \sum_1^\infty c_n \sin \frac{n\pi\theta}{\beta}, \qquad g(\theta) = \sum_1^\infty d_n \sin \frac{n\pi\theta}{\beta},$$

we see that

$$a_n r_1^{n\pi/\beta} + b_n r_1^{-n\pi/\beta} = c_n, \qquad a_n r_0^{n\pi/\beta} + b_n r_0^{-n\pi/\beta} = d_n,$$

and it is a simple matter to solve these equations simultaneously for a_n and b_n.

In a similar way we can solve the Dirichlet problem in an annulus

$$A = \Big\{(r\cos\theta, r\sin\theta) : r_0 \le r \le r_1,\ \theta \text{ arbitrary}\Big\},$$

namely,

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0 \quad \text{in } A, \qquad u(r_1, \theta) = f(\theta), \quad u(r_0, \theta) = g(\theta).$$

The boundary conditions at $\theta = 0$ and $\theta = \beta$ are now replaced by the requirement that u be 2π-periodic in θ. Thus, instead of (4.30) we ask for periodic solutions of $\Theta'' + \nu^2\Theta = 0$; this forces ν to be an integer and gives the eigenfunctions $e^{\pm in\theta}$ or $\cos n\theta$ and $\sin n\theta$, with the result that

$$u(r,\theta) = (a_0 + b_0 \log r) + \sum_{n=\pm1,\pm2,\dots} e^{in\theta}(a_n r^n + b_n r^{-n}). \tag{4.33}$$

Now the coefficients a_n and b_n are found by expanding the periodic functions f and g in their full Fourier series rather than a Fourier sine series.

Finally, we can let the inner radius r_0 tend to zero and consider the Dirichlet problem on a disc

$$D = \Big\{(x,y) : x^2 + y^2 \le r_1^2\Big\} = \Big\{(r\cos\theta,\ r\sin\theta) : r \le r_1\Big\},$$

that is,

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0 \quad \text{in } D, \qquad u(r_1,\theta) = f(\theta). \tag{4.34}$$

Here the inner boundary condition has disappeared, but there is still a condition to be satisfied at $r = 0$. Functions of the form (4.33) will satisfy Laplace's equation in the punctured disc $\{0 < r \le r_1\}$, but they will blow up at $r = 0$ unless all the terms involving $\log r$ or negative powers of r vanish. In other words, we impose the "boundary condition" on the product solutions $u = R\Theta$ obtained from (4.30) and (4.31) that they should be continuous at $r = 0$. The result is that (4.33) must be replaced by

$$u(r,\theta) = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta},$$

and the condition $u(r_1, \theta) = f(\theta)$ means that the numbers $c_n r_1^{|n|}$ are the Fourier coefficients of f.

From this we can derive a useful formula for the solution to (4.34) as an integral rather than a series. To simplify the calculation a bit, we shall take $r_1 = 1$; the reader may verify that for the general case one merely replaces r by r/r_1 in the following formulas. We recall that the Fourier coefficients of f are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)e^{-in\phi}\,d\phi.$$

If we substitute this into the formula for u, we obtain

$$u(r,\theta) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} r^{|n|}e^{in\theta}\int_{-\pi}^{\pi} f(\phi)e^{-in\phi}\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)P(r,\theta-\phi)\,d\phi$$

where $P(r,\theta)$ is the **Poisson kernel**:

$$P(r,\psi) = \sum_{-\infty}^{\infty} r^{|n|}e^{in\psi} = \sum_{0}^{\infty} r^n e^{in\psi} + \sum_{1}^{\infty} r^n e^{-in\psi}.$$

The series on the right are geometric series that converge nicely for $r < 1$. This fact justifies the interchange of integration and summation we have just performed, and it also allows one to sum the series in closed form:

$$\begin{aligned} P(r,\psi) &= \frac{1}{1-re^{i\psi}} + \frac{re^{-i\psi}}{1-re^{-i\psi}} = \frac{1-r^2}{(1-re^{i\psi})(1-re^{-i\psi})} \\ &= \frac{1-r^2}{1+r^2-2r\cos\psi}. \end{aligned}$$

In short, we have the **Poisson integral formula** for the solution of (4.34) (with $r_1 = 1$):

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} f(\phi)\,d\phi. \tag{4.35}$$

EXERCISES

Exercises 1–3 deal with the equation $\nabla^2 u = 0$ in the square

$$D = \{(x,y) : 0 \le x \le l,\ 0 \le y \le l\}.$$

1. Solve $\nabla^2 u = 0$ in D subject to the boundary conditions $u(x,0) = u(0,y) = u(l,y) = 0$, $u(x,l) = x(l-x)$. (Cf. Exercise 10, §2.4.)
2. Find the steady-state temperature in D if the sides $x = 0$ and $x = l$ are insulated, the side $y = 0$ is held at temperature zero, and the side $y = l$ is held at temperature $u(x,l) = x$.

3. Consider the *Neumann problem*

$$\nabla^2 u = 0 \text{ in } D, \qquad u_x(0,y) = u_x(l,y) = u_y(x,0) = 0, \quad u_y(x,l) = f(x).$$

(Thus the normal derivative of u on the boundary is prescribed.) Use Fourier cosine series to find a solution, if possible. Show that a solution exists only if $\int_0^l f(x)\,dx = 0$, in which case it contains an arbitrary constant.

4. Find the steady-state temperature in the semi-infinite strip $0 \le x \le l$, $0 \le y < \infty$ if $u(0,y) = u(l,y) = 0$ and $u(x,0) = f(x)$. (Hint: On physical grounds, $u(x,y)$ must be bounded in the strip.)

Exercises 5–8 deal with the equation $\nabla^2 u = 0$ in polar coordinates.

5. Suppose the inner side of the annulus $\{(r,\theta) : r_0 \le r \le 1\}$ is insulated and the outer side is held at temperature $u(1,\theta) = f(\theta)$.
   a. Find the steady-state temperature.
   b. What is the solution if $f(\theta) = 1 + 2\sin\theta$?
6. Let D be the unit disc $\{(r,\theta) : 0 \le r \le 1\}$. Let $P(r,\theta)$ be the Poisson kernel, and let $u(r,\theta)$ be the solution of the Dirichlet problem $\nabla^2 u = 0$ in D, $u(1,\theta) = f(\theta)$.
   a. Show that the value of u at the origin is $(2\pi)^{-1}\int_{-\pi}^{\pi} f(\theta)\,d\theta$. (This is the *mean value theorem* for harmonic functions: the value of a harmonic function at the center of a circle is the average of its values on the circle.)
   b. Show that $P(r,\theta) > 0$ and that $\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = 2\pi$ for all $r < 1$.
   c. Use part (b) to show that if $f(\theta) \le M$ for all θ, then $u(r,\theta) \le M$ for all θ and all $r < 1$. (This is the *maximum principle* for harmonic functions in a disc.)
7. Solve the following Dirichlet problem:

$$\nabla^2 u = 0 \quad \text{in } S = \Big\{(r,\theta) : 0 < r_0 \le r \le 1,\ 0 \le \theta \le \beta\Big\},$$
$$u(r_0,\theta) = u(1,\theta) = 0, \qquad u(r,0) = g(r), \quad u(r,\beta) = h(r).$$

(Cf. Exercise 10, §3.5.)

8. Consider the Dirichlet problem on the limiting case

$$S_0 = \Big\{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le \beta\Big\}$$

of the region S in Exercise 7.
   a. Solve: $\nabla^2 u = 0$ in S_0, $u(r,0) = u(r,\beta) = 0$ for $r < 1$, $u(1,\theta) = f(\theta)$. (This is problem (4.29) in the limiting case $r_0 = 0$, and the method used to solve (4.29) can be adapted. Note that the piece of the boundary $r = r_0$ has collapsed to a point, at which f has already been prescribed to be zero.)
   b. Try to solve the limiting case of Exercise 7: $\nabla^2 u = 0$ in S_0, $u(1,\theta) = 0$, $u(r,0) = g(r)$, $u(r,\beta) = h(r)$. (You won't succeed with the present methods. Separation of variables leads to the problem $(rf')' + (\lambda/r)f = 0$, $f(1) = 0$, which has *no* eigenfunctions in $L^2_{1/r}(0,1)$. This is a singular Sturm-Liouville problem whose solution requires integrals rather than infinite series; see Exercise 9, §7.4.)

4.5 Multiple Fourier series and applications

We have seen how Sturm-Liouville problems give rise to orthonormal bases for $L^2(a,b)$, but we have not yet seen any examples of orthonormal bases for $L^2(D)$ where D is a region in $\mathbf{R}^n$ with $n > 1$. However, for rectangular regions — that is, regions that are products of intervals — there is a simple way of building orthonormal bases out of the one-dimensional ones. Specifically, we have the following theorem.

Theorem 4.1. *Suppose $\{\phi_n\}_1^\infty$ is an orthonormal basis for $L^2(a,b)$ and $\{\psi_n\}_1^\infty$ is an orthonormal basis for $L^2(c,d)$. Let*

$$\chi_{mn}(x,y) = \phi_m(x)\psi_n(y).$$

Then $\{\chi_{mn}\}_{m,n=1}^\infty$ is an orthonormal basis for $L^2(D)$, where

$$D = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}.$$

Proof: Orthonormality is easy:

$$\begin{aligned}
\langle \chi_{mn}, \chi_{m'n'} \rangle &= \iint_D \chi_{mn}(x,y)\overline{\chi_{m'n'}(x,y)}\,dx\,dy \\
&= \int_c^d \int_a^b \phi_m(x)\psi_n(y)\overline{\phi_{m'}(x)\psi_{n'}(y)}\,dx\,dy \\
&= \left(\int_a^b \phi_m(x)\overline{\phi_{m'}(x)}\,dx\right)\left(\int_c^d \psi_n(y)\overline{\psi_{n'}(y)}\,dy\right) \\
&= \begin{cases} 1 & \text{if } m = m' \text{ and } n = n', \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

To prove completeness, we shall show that if $f \in L^2(D)$ and $\langle f, \chi_{mn} \rangle = 0$ for all m and n then $f = 0$. (The argument that follows is the truth but not quite the whole truth; we are glossing over some technical points about the workings of the Lebesgue integral. See Folland [25], §2.5, or Wheeden-Zygmund [56], Chapter 6.) Let

$$g_n(x) = \int_c^d f(x,y)\overline{\psi_n(y)}\,dy.$$

Then $g_n \in L^2(a,b)$, for by the Schwarz inequality and the fact that $\|\psi_n\| = 1$,

$$\begin{aligned}
\int_a^b |g_n(x)|^2 dx &\le \int_a^b \left(\int_c^d |f(x,y)|^2 dy\right)\left(\int_c^d |\psi_n(y)|^2 dy\right) dx \\
&= \int_a^b \int_c^d |f(x,y)|^2 dy\,dx < \infty.
\end{aligned}$$

Moreover,

$$\langle g_n, \phi_m \rangle = \int_a^b \int_c^d f(x,y)\overline{\psi_n(y)\phi_m(x)}\,dy\,dx = \langle f, \chi_{mn} \rangle = 0$$

for all m, so since $\{\phi_m\}$ is complete, we have $g_n(x) = 0$ for all n and (almost) all x. But $g_n(x) = \langle f(x,\cdot), \psi_n \rangle$, so the completeness of $\{\psi_n\}$ implies that $f(x,y) = 0$ for (almost) all x and y, that is, $f = 0$ as an element of $L^2(D)$. ∎

This theorem is valid (with essentially the same proof) in much greater generality than our statement of it. Here are four useful extensions of it:

(i) One can replace the intervals $[a,b]$ and $[c,d]$ by sets $A \subset \mathbf{R}^j$ and $B \subset \mathbf{R}^k$, in which case
$$D = A \times B = \Big\{(\mathbf{x},\mathbf{y}) \in \mathbf{R}^{j+k} : \mathbf{x} \in A,\ \mathbf{y} \in B\Big\}.$$

(ii) One can introduce weight functions. If $\{\phi_n\}$ is an orthonormal basis for $L^2_u(a,b)$ and $\{\psi_n\}$ is an orthonormal basis for $L^2_v(c,d)$, then $\{\chi_{mn}\}$ is an orthonormal basis for $L^2_w(D)$, where $w(x,y) = u(x)v(y)$.

(iii) One can consider products with more than two factors. For example, suppose that in addition to the data in the theorem we have an orthonormal basis $\{\theta_n\}$ for $L^2(\alpha,\beta)$. Then the products $\phi_l(x)\psi_m(y)\theta_n(z)$ form an orthonormal basis for $L^2(D)$, where

$$D = \Big\{(x,y,z) : a \le x \le b,\ c \le y \le d,\ \alpha \le z \le \beta\Big\}.$$

(iv) One can start with an orthonormal basis $\{\psi_n\}_{n=1}^{\infty}$ for $L^2(c,d)$, and for *each* n a different orthonormal basis $\{\phi_{m,n}\}_{m=1}^{\infty}$ for $L^2(a,b)$. Then $\{\chi_{mn}\}_{m,n=1}^{\infty}$ is an orthonormal basis for $L^2(D)$, where $\chi_{mn}(x,y) = \phi_{m,n}(x)\psi_n(y)$. This situation will turn up in Chapters 5 and 6; in particular, see Theorem 5.4 of §5.5.

One more comment about the theorem should be made. The assertion that $\{\chi_{mn}\}$ is a basis should mean that if $f \in L^2(D)$ then $f = \sum \langle f, \chi_{mn} \rangle \chi_{mn}$, but one must assign a precise meaning to such a double infinite series. In fact, there is no problem. One arranges the terms $\langle f, \chi_{m,n} \rangle \chi_{mn}$ into a single sequence in any way one wishes, and the resulting ordinary infinite series always converges in norm to f.

With this bit of machinery in hand, we can find useful series expansions for functions of two or more variables. The most basic example is the multiple Fourier series for periodic functions. Suppose, to be specific, that we wish to study functions $f(x,y)$ that are 1-periodic in each variable: $f(x+1,\,y) = f(x,y)$ and $f(x,\,y+1) = f(x,y)$. (Functions of this sort arise, for example, in the theory of crystal lattices in solid-state physics.) Such doubly periodic functions are competely determined by their restrictions to the unit square

$$S = [0,1] \times [0,1] = \Big\{(x,y) : 0 \le x, y \le 1\Big\}.$$

We already know that $\{e^{2\pi inx}\}_{-\infty}^{\infty}$ is an orthonormal basis for $L^2(0,1)$, so it follows that

$$\left\{\chi_{mn}(x,y) = e^{2\pi i(mx+ny)} : -\infty < m,n < \infty\right\}$$

is an orthonormal basis for $L^2(S)$. Thus we can expand any doubly periodic f that is square-integrable on S in a double Fourier series:

$$f = \sum_{m,n=-\infty}^{\infty} c_{mn}\chi_{mn}, \qquad c_{mn} = \langle f, \chi_{mn}\rangle = \int_0^1\int_0^1 f(x,y)e^{-2\pi i(mx+ny)}\,dx\,dy,$$

where the series converges (at least) in norm. (The reader should be warned that the question of pointwise convergence of multiple Fourier series is even more delicate than in the one-dimensional case, but norm convergence works equally easily in any number of dimensions.)

Similarly, we can form multiple Fourier cosine or sine series, or combine other orthonormal bases arising from Sturm-Liouville problems, to construct bases for functions on rectangular regions; and this procedure can be used to solve boundary value problems in dimensions $n > 1$. We illustrate this with some examples.

Example 1. We analyze the vibrations of an elastic membrane stretched across a rectangular frame. That is, we study the following boundary value problem for the wave equation in two space dimensions:

$$u_{tt} = c^2(u_{xx} + u_{yy}) \text{ for } 0 < x < l,\ 0 < y < L,$$
$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y),$$
$$u(0,y,t) = u(l,y,t) = u(x,0,t) = u(x,L,t) = 0.$$

It is pretty clear that we shall want to use a double Fourier sine series to solve this problem, but let us see explicitly how separation of variables leads to this construction. Neglecting the initial conditions for the moment, we look for product solutions $X(x)Y(y)T(t)$ of the wave equation in the rectangle with zero boundary values. The wave equation for such functions is

$$XYT'' = c^2(X''YT + XY''T), \quad \text{or} \quad \frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

The quantities on either side of the last equation must equal some constant, which we shall call $-\nu^2$, so $T'' + \nu^2c^2T = 0$ and

$$\frac{X''}{X} = -\frac{Y''}{Y} - \nu^2.$$

Now the quantities on either side of *this* equation must equal another constant, which we call $-\mu^2$. Taking the boundary conditions into account, we therefore have

$$X'' + \mu^2X = 0, \qquad X(0) = X(l) = 0,$$
$$Y'' + (\nu^2 - \mu^2)Y = 0, \qquad Y(0) = Y(L) = 0,$$

and hence

$$\mu^2 = \left(\frac{m\pi}{l}\right)^2, \qquad X(x) = \sin\frac{m\pi x}{l}, \qquad (m = 1, 2, 3, \ldots),$$
$$\nu^2 - \mu^2 = \left(\frac{n\pi}{L}\right)^2, \qquad Y(y) = \sin\frac{n\pi y}{L}, \qquad (n = 1, 2, 3, \ldots).$$

Finally, since $T'' + \nu^2 c^2 T = 0$, we have

$$T(t) = a_\nu \cos\nu ct + b_\nu \sin\nu ct \quad \text{where } \nu^2 = \mu^2 + (\nu^2 - \mu^2) = \left(\frac{m\pi}{l}\right)^2 + \left(\frac{n\pi}{L}\right)^2.$$

Thus we have the following solutions of the wave equation in the rectangle with zero boundary values:

$$u(x,y,t)$$
$$= \sum_{m,n=1}^{\infty} \sin\frac{m\pi x}{l}\sin\frac{n\pi y}{L}\left(a_{mn}\cos\pi ct\sqrt{\frac{m^2}{l^2}+\frac{n^2}{L^2}} + b_{mn}\sin\pi ct\sqrt{\frac{m^2}{l^2}+\frac{n^2}{L^2}}\right).$$

The coefficients a_{mn} and b_{mn} are determined by the initial conditions in the usual way: One expands f and g in their double Fourier sine series and matches coefficients with those of $u(x,y,0)$ and $u_t(x,y,0)$. Specifically,

$$u(x,y,0) = f(x,y) = \sum_{m,n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{l}\sin\frac{n\pi y}{L},$$

so that

$$a_{mn} = \frac{4}{lL}\int_0^l\int_0^L f(x,y)\sin\frac{m\pi x}{l}\sin\frac{n\pi y}{L}\,dy\,dx.$$

Qualitatively, the interesting feature here is the set of allowable frequencies of vibration, namely,

$$\left\{\pi c\sqrt{(m/l)^2 + (n/L)^2} : m, n = 1, 2, 3, \ldots\right\}.$$

In contrast to the case of 1-dimensional vibrations, these are not integer multiples of a fundamental frequency. For example, if $l = L = \pi c$, the lowest frequencies are $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{13}$, $\sqrt{17}$, and so forth. For this reason a rectangular membrane does not usually produce a musical sound as it vibrates. (The more commonly encountered case of a circular membrane will be studied in Chapter 5.)

Example 2. We consider heat flow in a rectangular solid

$$D = \left\{(x,y,z) : 0 \le x \le l_1,\ 0 \le y \le l_2,\ 0 \le z \le l_3\right\},$$

where the top and bottom faces are held at temperature zero and the other four faces are insulated. Thus, if the initial temperature is $f(x, y, z)$, the problem to be solved is

$$u_t = k(u_{xx} + u_{yy} + u_{zz}), \qquad u(x, y, z, 0) = f(x, y, z),$$
$$u(x, y, 0, t) = u(x, y, l_3, t) = 0,$$
$$u_x(0, y, z, t) = u_x(l_1, y, z, t) = u_y(x, 0, z, t) = u_y(x, l_2, z, t) = 0.$$

The process of separation of variables works here just as in the previous example, except that there is one more step (because there is one more variable), and the boundary conditions lead to Fourier cosine series in x and y rather than Fourier sine series. We leave it to the reader to work through the details; the upshot is that

$$u(x, y, z, t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=1}^{\infty} \epsilon_{n_1 n_2} a_{n_1 n_2 n_3} \exp\left[-\left(\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} + \frac{n_3^2}{l_3^2}\right)\pi^2 kt\right] \times \cos\frac{n_1 \pi x}{l_1} \cos\frac{n_2 \pi y}{l_2} \sin\frac{n_3 \pi z}{l_3}.$$

Here $\epsilon_{n_1 n_2}$ equals 1 when n_1 and n_2 are both nonzero, $\frac{1}{2}$ when one of n_1 and n_2 is zero but not both, and $\frac{1}{4}$ when $n_1 = n_2 = 0$ (this is to account for the usual factor of $\frac{1}{2}$ in the constant term of a Fourier cosine series), and the coefficients $a_{n_1 n_2 n_3}$ are the Fourier coefficients of the initial temperature:

$$a_{n_1 n_2 n_3} = \frac{8}{l_1 l_2 l_3} \int_0^{l_1} \int_0^{l_2} \int_0^{l_3} f(x, y, z) \cos\frac{n_1 \pi x}{l_1} \cos\frac{n_2 \pi y}{l_2} \sin\frac{n_3 \pi z}{l_3}\, dz\, dy\, dx.$$

Example 3. Suppose the rectangular box D of Example 2 is filled with a distribution of electric charge with density $\rho(x, y, z)$, and the faces of the box are grounded so that their electrostatic potential is zero. What is the potential inside the box? What we want is the solution of

$$u_{xx} + u_{yy} + u_{zz} = -4\pi \rho(x, y, z) \text{ in } D,$$
$$u(0, y, z) = u(l_1, y, z) = u(x, 0, z) = u(x, l_2, z) = u(x, y, 0) = u(x, y, l_3) = 0.$$

Here Technique 2 of §4.1 is effective. Namely, the zero boundary conditions suggest the use of Fourier sine series in each variable, so we expand u in such a series:

$$u(x, y, z) = \sum_{n_1, n_2, n_3=1}^{\infty} b_{n_1 n_2 n_3} \sin\frac{n_1 \pi x}{l_1} \sin\frac{n_2 \pi y}{l_2} \sin\frac{n_3 \pi z}{l_3}.$$

Computing $\nabla^2 u$ by termwise differentiation, we find

$$\nabla^2 u(x, y, z) = -\sum_{n_1, n_2, n_3=1}^{\infty} \left(\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} + \frac{n_3^2}{l_3^2}\right) \pi^2 b_{n_1 n_2 n_3} \sin\frac{n_1 \pi x}{l_1} \sin\frac{n_2 \pi y}{l_2} \sin\frac{n_3 \pi z}{l_3}.$$

This must be the multiple sine series for $-4\pi\rho$, so we can solve immediately for the coefficients $b_{n_1n_2n_3}$:

$$b_{n_1n_2n_3} = \frac{32}{\pi l_1 l_2 l_3}\left(\frac{n_1^2}{l_1^2}+\frac{n_2^2}{l_2^2}+\frac{n_3^2}{l_3^2}\right)^{-1} \times \int_0^{l_1}\int_0^{l_2}\int_0^{l_3} \rho(x,y,z)\sin\frac{n_1\pi x}{l_1}\sin\frac{n_2\pi y}{l_2}\sin\frac{n_3\pi z}{l_3}\,dz\,dy\,dx.$$

This formal procedure for solving the problem can be justified easily if we impose conditions on ρ so that its Fourier coefficients tend rapidly to zero (e.g., ρ and its first few derivatives should vanish on the boundary of the box). It can also be justified by more sophisticated methods just under the condition that $\rho \in L^2(D)$.

EXERCISES

1. Show that if $v(x,t)$ and $w(y,t)$ are solutions of the 1-dimensional heat equation ($v_t = kv_{xx}$ and $w_t = kw_{yy}$), then $u(x,y,t) = v(x,t)w(y,t)$ satisfies the 2-dimensional heat equation. Can you generalize to 3 dimensions? Is the same result true for solutions of the wave equation?
2. Redo Example 1 in the text for the damped wave equation $u_{tt} + 2ku_t = c^2(u_{xx} + u_{yy})$. (Cf. Exercise 6, §4.2.)
3. Solve the wave equation (with general initial conditions) for a rectangular membrane if one pair of opposite edges is held fixed ($u(0,y,t) = u(l,y,t) = 0$) and the other pair is free ($u_y(x,0,t) = u_y(x,L,t) = 0$). How do the frequencies compare with those of Example 1?
4. Let D be the rectangular box of Example 2. Suppose the faces $z = 0$ and $z = l_3$ are insulated, and the other four faces are kept at temperature zero. Find the temperature $u(x,y,z,t)$ given that $u(x,y,z,0) = f(x,y)$. (Hint: Since f is independent of z and the z-faces are insulated, you can treat this as a 2-dimensional problem.)
5. In Example 3, suppose $l_1 = l_2 = l_3 = \pi$ and $\rho(x,y,z) = x$. What is the potential u?
6. Consider a cubic crystal lattice in which the charge density $\rho(x,y,z)$ is $2l$-periodic in each variable. (We suppose that the lattice extends infinitely in all directions; this is reasonable if its actual size is very large in comparison with the length scale being studied.) Use Fourier series to find a periodic solution of $\nabla^2 u = -4\pi\rho$, assuming that the net charge in any cube of side $2l$ is zero. (This assumption is generally valid in practice. Why is it needed mathematically?)

CHAPTER 5
BESSEL FUNCTIONS

The 2-dimensional wave equation in polar coordinates is

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}).$$

(See Appendix 4 for the calculation of the Laplace operator in polar coordinates.) If we apply separation of variables by taking $u = R(r)\Theta(\theta)T(t)$, the wave equation becomes

$$\frac{T''}{c^2T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta}.$$

Both sides must equal a constant, which we shall call $-\mu^2$. Setting the expression on the right equal to $-\mu^2$ and multiplying through by r^2, we obtain

$$\frac{r^2R''}{R} + \frac{rR'}{R} + \mu^2r^2 = -\frac{\Theta''}{\Theta}.$$

Here both sides must equal another constant, which we call ν^2, so we arrive at the ordinary differential equations

$$T'' + c^2\mu^2T = 0 \quad \text{and} \quad \Theta'' + \nu^2\Theta = 0,$$

which are familiar enough, and

$$r^2R''(r) + rR'(r) + (\mu^2r^2 - \nu^2)R(r) = 0, \tag{5.1}$$

which is new. Equation (5.1) can be simplified a bit by the change of variable $x = \mu r$. (It is not yet clear whether we want μ to be real or imaginary, but at this point it doesn't matter; there is no harm in letting x be a complex variable.) That is, we substitute

$$R(r) = f(\mu r), \quad R'(r) = \mu f'(\mu r), \quad R''(r) = \mu^2 f''(\mu r), \quad \text{and} \quad r = x/\mu$$

into (5.1), obtaining

$$\left(\frac{x}{\mu}\right)^2 \mu^2 f''(x) + \frac{x}{\mu}\mu f'(x) + (x^2 - \nu^2)f(x) = 0,$$

or

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0. \tag{5.2}$$

This is **Bessel's equation of order** ν. It and its variants arise in many problems in physics and engineering, particularly where some sort of circular symmetry is involved. For this reason, its solutions are sometimes called **cylinder functions**, but we shall use the more common term **Bessel functions**. This chapter is an exposition of the basic properties of Bessel functions and some of their applications. Further information on Bessel functions can be found in Erdélyi et al. [21], Hochstadt [30], Lebedev [36], and especially the classic treatise of Watson [55].

5.1 Solutions of Bessel's equation

In this section we construct solutions of Bessel's equation (5.2) by means of power series. For the time being, the variable x and the parameter ν can be arbitrary complex numbers, although for most applications they will both be real and nonnegative. We shall make occasional references to the complex-variable properties of Bessel functions, but the reader who wishes to ignore them will not miss much as far as the material in this chapter is concerned.

At the outset, we note that (5.2) is unchanged if ν is replaced by $-\nu$, so we can take $\mathrm{Re}(\nu) \geq 0$ (in particular, $\nu \geq 0$ when ν is real) whenever it is convenient.

The differential equation (5.2) has a regular singular point at $x = 0$, so we expect to find solutions of the form

$$f(x) = \sum_0^\infty a_j x^{j+b} \qquad (a_0 \neq 0), \tag{5.3}$$

where the exponent b and the coefficients a_j are to be determined. If we substitute (5.3) into (5.2), we obtain

$$\sum_0^\infty a_j \Big[(j+b)(j+b-1)x^{j+b} + (j+b)x^{j+b} + x^{j+b+2} - \nu^2 x^{j+b}\Big] = 0. \tag{5.4}$$

We separate out the terms x^{j+b+2} and relabel the index of summation,

$$\sum_0^\infty a_j x^{j+b+2} = a_0 x^{2+b} + a_1 x^{3+b} + a_2 x^{4+b} + \cdots = \sum_2^\infty a_{j-2} x^{j+b},$$

thus transforming (5.4) into

$$\sum_0^\infty \Big[(j+b)^2 - \nu^2\Big] a_j x^{j+b} + \sum_2^\infty a_{j-2} x^{j+b} = 0.$$

Now, a power series can vanish identically only when all of its coefficients are zero, so we obtain the following sequence of equations:

$$\text{for } j = 0, \quad (b^2 - \nu^2)a_0 = 0, \tag{5.5}$$

$$\text{for } j = 1, \quad \left[(1+b)^2 - \nu^2\right]a_1 = 0, \tag{5.6}$$

$$\text{for } j \geq 2, \quad \left[(j+b)^2 - \nu^2\right]a_j + a_{j-2} = 0. \tag{5.7}$$

Since we assumed that $a_0 \neq 0$, equation (5.5) forces $b = \pm\nu$, and for the time being we take $b = \nu$. Then equation (5.6) becomes $(2\nu + 1)a_1 = 0$, so we must have $a_1 = 0$ except when $\nu = -\frac{1}{2}$; even when $\nu = -\frac{1}{2}$ it is consistent to take $a_1 = 0$, and we do so. Next, equation (5.7) with $b = \nu$ says that

$$a_j = -\frac{a_{j-2}}{(j+\nu)^2 - \nu^2} = -\frac{a_{j-2}}{j(j+2\nu)}. \tag{5.8}$$

From this recursion formula we can solve for all the even-numbered coefficients in terms of a_0:

$$a_2 = -\frac{a_0}{2(2+2\nu)}, \qquad a_4 = -\frac{a_2}{4(4+2\nu)} = \frac{a_0}{2\cdot 4(2+2\nu)(4+2\nu)},$$

and in general,

$$\begin{aligned} a_{2k} &= \frac{(-1)^k a_0}{2\cdot 4\cdots(2k)(2+2\nu)(4+2\nu)\cdots(2k+2\nu)} \\ &= \frac{(-1)^k a_0}{2^{2k}k!(1+\nu)(2+\nu)\cdots(k+\nu)}. \end{aligned} \tag{5.9}$$

In the same way, we obtain all of the odd-numbered coefficients in terms of a_1; but $a_1 = 0$, and hence

$$a_{2k+1} = 0.$$

The only time when this procedure runs into difficulties is when ν is a negative integer or half-integer. If $\nu = -n$, the numbers a_{2k} in (5.9) are ill-defined for $k \geq n$ because of a zero factor in the denominator, so in this case we do not obtain a solution. If $\nu = -n/2$ with n odd, the recursion formula (5.8) has a zero in the denominator for $j = n$, so our derivation of the string of equations $0 = a_1 = a_3 = \cdots$ breaks down at this point. However, if we rewrite (5.8) in the original form (5.7), then for $j = n$ it says that $0 \cdot a_n = a_{n-2}$. Since we already have $a_{n-2} = 0$, it is still consistent to take $a_n = 0$, and we do so. (We could take a_n to be something else; then we would get a different solution of the differential equation.)

In short, except when ν is a negative integer we have the solutions

$$f(x) = a_0 \sum_0^\infty \frac{(-1)^k x^{2k+\nu}}{2^{2k}k!(1+\nu)(2+\nu)\cdots(k+\nu)}$$

to Bessel's equation. It remains to pick the constant a_0, and the standard choice is

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}.$$

(See Appendix 3 for a discussion of the gamma function.) Since the functional equation $\Gamma(z+1) = z\Gamma(z)$ implies that

$$\Gamma(k+\nu+1) = (k+\nu)\cdots(1+\nu)\Gamma(\nu+1),$$

this choice of a_0 makes $f(x)$ equal to

$$J_\nu(x) = \sum_0^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu} \tag{5.10}$$

A simple application of the ratio test shows that this series is absolutely convergent for all $x \neq 0$ (and also for $x = 0$ when $\operatorname{Re}(\nu) > 0$ or $\nu = 0$). The function $J_\nu(x)$ thus defined is called the **Bessel function** (of the first kind) **of order** ν.

$J_\nu(x)$ is real when $x > 0$ and ν is real (the case we shall mainly be interested in). It tends to 0 as $x \to 0$ whenever $\operatorname{Re}(\nu) > 0$ and blows up as $x \to 0$ whenever $\operatorname{Re}(\nu) < 0$ and ν is not an integer. If we consider x as a complex variable, $J_\nu(x)$ is multivalued when ν is not an integer; to make a well-defined function we shall always take the principal branch of $(x/2)^\nu$ in (5.10). (That is, $(x/2)^\nu = e^{\nu \log(x/2)}$ where $-\pi < \operatorname{Im} \log(x/2) \leq \pi$.) However, $x^{-\nu} J_\nu(x)$ is an entire analytic function of x for any ν.

When ν is a nonnegative integer n, we can use the fact that $j! = \Gamma(j+1)$ to write

$$J_n(x) = \sum_0^\infty \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k+n} \qquad (n = 0, 1, 2, \ldots).$$

Also, it is to be observed that the definition (5.10) makes sense when ν is a negative integer, even though the formulas leading to it do not! Indeed, we recall that $1/\Gamma(z) = 0$ when $z = 0, -1, -2, \ldots$; hence if $\nu = -n$, we have $1/\Gamma(k+\nu+1) = 0$ for $k = 0, 1, \ldots, n-1$. Thus, the first n terms in the series (5.10) vanish, and by setting $k = j + n$ in (5.10) we find that

$$J_{-n}(x) = \sum_{k=n}^\infty \frac{(-1)^k}{k!(k-n)!} \left(\frac{x}{2}\right)^{2k-n} = \sum_{j=0}^\infty \frac{(-1)^{j+n}}{j!(j+n)!} \left(\frac{x}{2}\right)^{2j+n},$$

or

$$J_{-n}(x) = (-1)^n J_n(x). \tag{5.11}$$

(In effect, we have compensated for the zero in the denominators of (5.9) by taking the leading coefficient a_0 equal to 0.)

We arrived at (5.10) by taking $b = \nu$ in the recursion formula (5.7). It is easily checked that if we take $b = -\nu$ instead, we get the same results with ν replaced by $-\nu$ throughout; in other words, we end up with $J_{-\nu}(x)$. Thus, we have two solutions J_ν and $J_{-\nu}$ of Bessel's equation (5.2). When ν is not an integer they are clearly linearly independent, since

$$J_\nu(x) \approx \frac{x^\nu}{2^\nu \Gamma(\nu+1)}, \qquad J_{-\nu}(x) \approx \frac{x^{-\nu}}{2^{-\nu}\Gamma(-\nu+1)} \quad \text{for } x \text{ near } 0.$$

In this case the general solution of (5.2) is a linear combination of J_ν and $J_{-\nu}$.

However, if ν is an integer then $J_{-\nu} = (-1)^\nu J_\nu$ by (5.11), so in this case we are still lacking a second independent solution. The standard way out of this difficulty is as follows. For ν not an integer, we define the **Weber function** or **Bessel function of the second kind** Y_ν by

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{5.12}$$

Y_ν is a linear combination of J_ν and $J_{-\nu}$, so it satisfies Bessel's equation (5.2). Also, since the coefficient of $J_{-\nu}(x)$ in (5.12) is nonzero, J_ν and Y_ν are linearly independent, and we may use them (rather than J_ν and $J_{-\nu}$) as a basis for solutions of (5.2). Now if we take ν to be an integer in (5.12), the expression on the right turns into the indeterminate 0/0, by (5.11) and the fact that $\cos n\pi = (-1)^n$. However, it can be shown that the limit as ν approaches an integer of $Y_\nu(x)$ exists and is finite for all $x \neq 0$, and that the function

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

thus defined is a solution of Bessel's equation. (See, for example, Lebedev [36], §5.4.) In fact, one can calculate $Y_n(x)$ from (5.12), (5.10), and l'Hôpital's rule. We shall not present the details since the explicit formula for Y_n will be of no particular use to us. (However, see Exercises 3–5.) The most important feature of Y_n is its asymptotic behavior as $x \to 0$, which for $n \geq 0$ is given by

$$Y_n(x) \approx -\frac{(n-1)!}{\pi}\left(\frac{x}{2}\right)^{-n} \quad \text{as } x \to 0 \qquad (n = 1, 2, 3, \ldots),$$
$$Y_0(x) \approx \frac{2}{\pi}\log\frac{x}{2} \quad \text{as } x \to 0.$$

(It follows from (5.11) and (5.12) that $Y_{-n} = (-1)^n Y_n$, so the case $n < 0$ is also covered.) In particular, $Y_n(x)$ blows up as $x \to 0$, whereas $J_n(x)$ remains bounded; so Y_n and J_n are linearly independent and form a basis for all solutions of Bessel's equation of order n. See Figure 5.1.

One may wonder why Y_ν was chosen as the particular linear combination (5.12) of J_ν and $J_{-\nu}$. If the only object had been to find an expression that gives a second solution for integer values of ν, many other formulas would have worked equally well; indeed, several variants of Y_ν are found in the literature. (See Watson [55], §3.5.) The answer has to do with the behavior of $J_\nu(x)$ and $Y_\nu(x)$ as $x \to \infty$, a matter that will be discussed in §5.3.

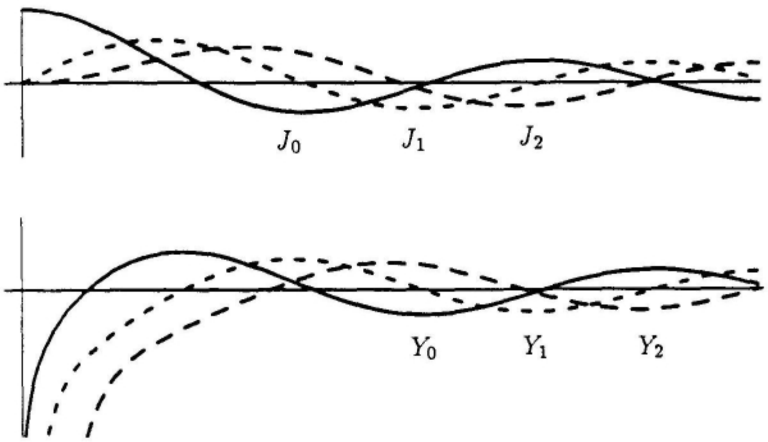


FIGURE 5.1. Graphs of some Bessel functions on the interval $0 \le x \le 10$. Top: J_0 (solid), J_1 (short dashes), and J_2 (long dashes). Bottom: Y_0 (solid), Y_1 (short dashes), and Y_2 (long dashes).

EXERCISES

1. Let f_1 and f_2 be solutions of Bessel's equation of order ν, and let W denote their Wronskian $f_1 f_2' - f_1' f_2$.
   a. Use Bessel's equation to show that $W'(x) = -W(x)/x$ and hence show that $W(x) = C/x$ for some constant C.
   b. Show that if $f_1 = J_\nu$ and $f_2 = J_{-\nu}$, then $W(x) = -2 \sin \nu\pi/\pi x$. (Hint: Consider the limiting behavior of $J_\nu(x)$ and $J_{-\nu}(x)$ as $x \to 0$, and use the fact that $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin \nu\pi$.)
   c. Show that if $f_1 = J_\nu$ and $f_2 = Y_\nu$ then $W(x) = 2/\pi x$.
2. Deduce from (5.11) and (5.12) that $Y_{-n} = (-1)^n Y_n$ when n is an integer.
3. When $\nu > 0$ and ν is not an integer, $Y_\nu(x)$ is given by a power series whose lowest-order term is $c_\nu (x/2)^{-\nu}$. What is the constant c_ν? Show that if n is a positive integer, $\lim_{\nu \to n} c_\nu = -(n-1)!/\pi$. (Hint: $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin \nu\pi$.)
4. Show that when n is an integer,
$$Y_n(x) = \frac{1}{\pi}\left[\frac{\partial J_\nu}{\partial \nu} + (-1)^{n+1}\frac{\partial J_{-\nu}}{\partial \nu}\right]_{\nu=n}.$$
5. Let $\psi(z) = \Gamma'(z)/\Gamma(z)$. Use Exercise 4 to show that
$$Y_0(x) = \frac{2}{\pi} J_0(x) \log \frac{x}{2} + \frac{2}{\pi}\sum_0^\infty \frac{(-1)^{j+1}\psi(j+1)}{(j!)^2}\left(\frac{x}{2}\right)^{2j}.$$
(For the ambitious reader: Find the analogous formula for Y_n, $n > 0$. The answer can be found in Lebedev [36], §5.5.)

5.2 Bessel function identities

There is a vast assortment of formulas relating Bessel functions to one another and to various other special functions; some are algebraic relations, and others involve integrals or infinite series. In this section we discuss a few of the most elementary and useful of them.

To begin with, there is a nice set of algebraic identities relating J_ν and its derivative to the "adjacent" functions $J_{\nu-1}$ and $J_{\nu+1}$:

The Recurrence Formulas. *For all x and ν,*

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = -x^{-\nu}J_{\nu+1}(x), \tag{5.13}$$

$$\frac{d}{dx}\left[x^{\nu}J_\nu(x)\right] = x^{\nu}J_{\nu-1}(x), \tag{5.14}$$

$$xJ_\nu'(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x), \tag{5.15}$$

$$xJ_\nu'(x) + \nu J_\nu(x) = xJ_{\nu-1}(x), \tag{5.16}$$

$$xJ_{\nu-1}(x) + xJ_{\nu+1}(x) = 2\nu J_\nu(x), \tag{5.17}$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x). \tag{5.18}$$

Proof: To prove (5.13) we use the power series (5.10) for $J_\nu(x)$:

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = \frac{d}{dx}\sum_0^\infty \frac{(-1)^k x^{2k}}{2^{2k+\nu}k!\Gamma(\nu+k+1)} = \sum_1^\infty \frac{(-1)^k(2k)x^{2k-1}}{2^{2k+\nu}k!\Gamma(\nu+k+1)}$$

$$= \sum_1^\infty \frac{(-1)^k x^{2k-1}}{2^{2k+\nu-1}(k-1)!\Gamma(\nu+k+1)}.$$

We relabel the index k in the last sum as $k+1$, thus obtaining

$$\sum_0^\infty \frac{(-1)^{k+1}x^{2k+1}}{2^{2k+\nu+1}k!\Gamma(\nu+k+2)} = -x^{-\nu}\sum_0^\infty \frac{(-1)^k x^{2k+\nu+1}}{2^{2k+\nu+1}k!\Gamma(\nu+k+2)} = -x^{-\nu}J_{\nu+1}(x).$$

The proof of (5.14) is similar:

$$\frac{d}{dx}\left[x^{\nu}J_\nu(x)\right] = \frac{d}{dx}\sum_0^\infty \frac{(-1)^k x^{2k+2\nu}}{2^{2k+\nu}k!\Gamma(k+\nu+1)} = \sum_0^\infty \frac{(-1)^k(2k+2\nu)x^{2k+2\nu-1}}{2^{2k+\nu}k!\Gamma(k+\nu+1)}$$

$$= x^\nu\sum_0^\infty \frac{(-1)^k x^{2k+\nu-1}}{2^{2k+\nu-1}k!\Gamma(k+\nu)} = x^\nu J_{\nu-1}(x).$$

Next, performing the indicated differentiations in (5.13) and (5.14), we obtain

$$(-\nu)x^{-\nu-1}J_\nu(x) + x^{-\nu}J_\nu'(x) = -x^{-\nu}J_{\nu+1}(x),$$

$$\nu x^{\nu-1}J_\nu(x) + x^{\nu}J_\nu'(x) = x^{\nu}J_{\nu-1}(x).$$

(5.15) and (5.16) are obtained by multiplying these equations through by $x^{\nu+1}$ and $x^{-\nu+1}$, respectively. Finally, (5.17) and (5.18) follow by subtracting and adding (5.15) and (5.16). ∎

As a first application of these formulas, we shall show that the Bessel functions of half-integer order can be expressed in terms of familiar elementary functions. To start with, consider $J_{-1/2}(x)$. Since

$$2^k k! = 2^k(1 \cdot 2 \cdot 3 \cdots k) = 2 \cdot 4 \cdot 6 \cdots (2k)$$

and

$$\begin{aligned} 2^k \Gamma(k + \tfrac{1}{2}) &= 2^k (k - \tfrac{1}{2})(k - \tfrac{3}{2}) \cdots (\tfrac{1}{2})\Gamma(\tfrac{1}{2}) \\ &= (2k-1)(2k-3)\cdots(1)\sqrt{\pi}, \end{aligned}$$

we have

$$J_{-1/2}(x) = \sum_0^\infty \frac{(-1)^k x^{2k-(1/2)}}{2^{-1/2}\left[2^k k!\right]\left[2^k \Gamma(k+\frac{1}{2})\right]} = \left(\frac{2}{\pi x}\right)^{1/2} \sum_0^\infty \frac{(-1)^k x^{2k}}{(2k)!}$$

But the last series is just the Taylor series of $\cos x$. This, together with a similar calculation for $J_{1/2}(x)$ (see Exercise 1), shows that

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x. \tag{5.19}$$

It now follows by repeated application of the recurrence formula (5.17) that *whenever* $\nu - \frac{1}{2}$ *is an integer,*

$$J_\nu(x) = x^{-1/2}\left[P_\nu(x)\cos x + Q_\nu(x)\sin x\right]$$

where P_ν *and* Q_ν *are rational functions.* For example, taking $\nu = \frac{1}{2}$ in (5.17), we find that

$$J_{3/2}(x) = x^{-1}J_{1/2}(x) - J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{\sin x}{x} - \cos x\right).$$

It is to be emphasized that the Bessel functions of half-integer order are the *only* ones that are elementary functions.

Our next group of results concerns the Bessel functions of integer order. We recall that if $\{a_n\}$ is a sequence of numbers, the *generating function* for a_n is the power series $\sum a_n z^n$.

The Generating Function for $J_n(x)$. *For all* x *and all* $z \neq 0$,

$$\sum_{-\infty}^{\infty} J_n(x) z^n = \exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right]. \tag{5.20}$$

Proof: We begin by observing that

$$\exp\frac{xz}{2} = \sum_0^\infty \frac{z^j}{j!}\left(\frac{x}{2}\right)^j, \qquad \exp\frac{-x}{2z} = \sum_0^\infty \frac{(-1)^k}{z^k k!}\left(\frac{x}{2}\right)^k.$$

Since these series are absolutely convergent, they can be multiplied together and the terms in the resulting double series summed in any order:

$$\exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right] = \sum_{j,k=0}^\infty \frac{(-1)^k z^{j-k}}{j!k!}\left(\frac{x}{2}\right)^{j+k}.$$

We sum this series by first adding up all the terms involving a given power z^n of z and then summing over n. That is, we set $j - k = n$ or $j = k + n$ and obtain (with the understanding that $1/(k+n)! = 1/\Gamma(k+n+1) = 0$ when $k + n < 0$)

$$\exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right] = \sum_{n=-\infty}^\infty \left[\sum_{k=0}^\infty \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\right] z^n$$

$$= \sum_{-\infty}^\infty J_n(x) z^n. \qquad \blacksquare$$

In (5.20), z can be any nonzero complex number. In particular, we can take $z = e^{i\theta}$, in which case $\frac{1}{2}(z - z^{-1}) = i \sin\theta$, so that

$$e^{ix\sin\theta} = \sum_{-\infty}^\infty J_n(x) e^{in\theta}. \tag{5.21}$$

But the expression on the left is a 2π-periodic function of θ, and the expression on the right is visibly a Fourier series! Hence, the coefficients $J_n(x)$ in the series must be given by the usual formula for Fourier coefficients, namely,

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta - in\theta}\, d\theta. \tag{5.22}$$

Here is a new formula for J_n, quite different from its original definition as a power series and in some respects more useful. For instance, it shows immediately, what is not at all evident from the power series, that $|J_n(x)| \le 1$ for all real x:

$$|J_n(x)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |e^{ix\sin\theta - in\theta}|\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta = 1 \qquad (x \in \mathbf{R}).$$

In fact, the same is true of all the derivatives of $J_n(x)$, for differentiation of (5.22) yields

$$J_n^{(k)}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (i\sin\theta)^k e^{ix\sin\theta - in\theta}\, d\theta.$$

There are several other formulas for J_n that are equivalent to (5.22). We collect them in a theorem.

Bessel's Integral Formulas. *For any $x \in \mathbf{C}$ and any integer n,*

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta - in\theta}\,d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)\,d\theta. \tag{5.23}$$

Moreover,

$$\begin{aligned} J_n(x) &= \frac{2}{\pi}\int_0^{\pi/2}\cos(x\sin\theta)\cos n\theta\,d\theta \quad \textit{if } n \textit{ is even;} \\ J_n(x) &= \frac{2}{\pi}\int_0^{\pi/2}\sin(x\sin\theta)\sin n\theta\,d\theta \quad \textit{if } n \textit{ is odd.} \end{aligned} \tag{5.24}$$

Proof: The change of variable $\theta \to -\theta$ in (5.22) leads to

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\sin\theta + in\theta}\,d\theta.$$

Adding this equation to (5.22) and dividing by 2, we obtain

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta - n\theta)\,d\theta.$$

Here the integrand is an even function of θ, so the integral from $-\pi$ to π is twice the integral from 0 to π; hence (5.23) follows.

Next, in (5.23) we make the change of variable $\theta \to \pi - \theta$ and use the fact that $\sin(\pi - \theta) = \sin\theta$ and $\cos(\phi - n\pi) = (-1)^n\cos\phi$ to obtain

$$(-1)^n J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta + n\theta)\,d\theta.$$

We now add or subtract this equation to (5.23), depending on whether n is even or odd, and use the identities

$$\begin{aligned} \cos(a-b) + \cos(a+b) &= 2\cos a\cos b, \\ \cos(a-b) - \cos(a+b) &= 2\sin a\sin b, \end{aligned}$$

with the result that

$$\begin{aligned} J_n(x) &= \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos n\theta\,d\theta \qquad (n \text{ even}), \\ J_n(x) &= \frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin n\theta\,d\theta \qquad (n \text{ odd}). \end{aligned}$$

The integrands in these equations are invariant under the substitution $\theta \to \pi - \theta$, i.e., symmetric about $\theta = \pi/2$, so the integrals from 0 to π are twice the integrals from 0 to $\pi/2$. This proves (5.24). ∎

If one is interested in calculating $J_n(x)$ numerically, the power series (5.10) is effective for small values of x. However, Bessel's integrals (evaluated by a numerical integration scheme such as Simpson's rule) are much more efficient when x is reasonably large. Similar but more complicated formulas exist for $J_\nu(x)$ when ν is not an integer.

EXERCISES

1. Show that $J_{1/2}(x) = \sqrt{2/\pi x}\,\sin x$.

Use the recurrence formulas to prove the identities in Exercises 2–7.

2. $J_{-3/2}(x) = -\sqrt{2/\pi x}\,(x^{-1}\cos x + \sin x)$
3. $x^2 J_\nu''(x) = (\nu^2 - \nu - x^2)J_\nu(x) + xJ_{\nu+1}(x)$
4. $\int_0^x sJ_0(s)\,ds = xJ_1(x)$ and $\int_0^x J_1(s)\,ds = 1 - J_0(x)$
5. $\int_0^x s^2 J_1(s)\,ds = 2xJ_1(x) - x^2 J_0(x)$
6. $\int_0^x J_3(s)\,ds = 1 - J_2(x) - 2x^{-1}J_1(x)$ (Hint: $J_3(s) = s^2 \cdot s^{-2}J_3(s)$.)
7. $(\nu+3)x^2 J_\nu(x) + 2(\nu+2)\left[x^2 - 2(\nu+1)(\nu+3)\right]J_{\nu+2}(x) + (\nu+1)x^2 J_{\nu+4}(x) = 0$ (Hint: Use (5.17) to express $J_{\nu+2}$ in terms of $J_{\nu+1}$ and $J_{\nu+3}$; then use (5.17) on the latter functions.)
8. Prove the reduction formula

$$\int_0^x s^n J_0(s)\,ds = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2\int_0^x s^{n-2}J_0(s)\,ds.$$

(Hint: Integrate by parts, using the facts that $(xJ_1)' = xJ_0$ and $J_0' = -J_1$.) Use this formula to show that

$$\int_0^x s^3 J_0(s)\,ds = (x^3 - 4x)J_1(x) + 2x^2 J_0(x),$$
$$\int_0^x s^5 J_0(s)\,ds = x(x^2-8)^2 J_1(x) + 4x^2(x^2-8)J_0(x).$$

Exercises 9–11 are applications of formulas (5.11) and (5.21).

9. Show that for all x,

$$J_0(x) + 2\sum_1^\infty J_{2n}(x) = 1, \qquad \sum_1^\infty (2n-1)J_{2n-1}(x) = \frac{x}{2}.$$

(Hint: To obtain the second formula, differentiate both sides of (5.21).)

10. Show that for each fixed x, $\lim_{n\to\infty} n^k J_n(x) = 0$ for all k. (Hint: Theorem 2.6, §2.3.)
11. Show that for all real x,

$$J_0(x)^2 + 2\sum_1^\infty J_n(x)^2 = 1.$$

(Hint: Parseval's equation.) Deduce that $|J_0(x)| \le 1$ and $|J_n(x)| \le 2^{-1/2}$ for $n > 0$.

12. Verify directly from the formula $J_0(x) = (2/\pi)\int_0^{\pi/2}\cos(x\sin\theta)\,d\theta$ that J_0 satisfies Bessel's equation of order zero.
13. Deduce from (5.24) that for $n = 0, 1, 2, \ldots$,

$$J_{2n}(x) = (-1)^n \frac{2}{\pi}\int_0^{\pi/2} \cos(x\cos\theta)\cos 2n\theta\,d\theta,$$

$$J_{2n+1}(x) = (-1)^n \frac{2}{\pi}\int_0^{\pi/2} \sin(x\cos\theta)\cos(2n-1)\theta\,d\theta.$$

14. (Poisson's integral for J_ν) Show that if $\mathrm{Re}(\nu) > -\frac{1}{2}$,

$$J_\nu(x) = \frac{x^\nu}{2^\nu\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\nu-(1/2)}e^{ixt}\,dt.$$

(Hint: Write $e^{ixt} = \sum_0^\infty (ixt)^j/j!$ and integrate the series term by term. The resulting integrals can be evaluated in terms of the beta function; see Appendix 3.) Deduce that

$$J_\nu(x) = \frac{x^\nu}{2^\nu\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\int_{-\pi/2}^{\pi/2} e^{ix\sin\theta}\cos^{2\nu}\theta\,d\theta.$$

5.3 Asymptotics and zeros of Bessel functions

The power series (5.10) that defines $J_\nu(x)$ readily yields precise information about $J_\nu(x)$ when x is near 0 but is of little value when x is large. The integral formulas of §5.2 are somewhat more helpful in this regard, but we still have only a rather vague idea of how $J_\nu(x)$ behaves as $x \to \infty$. One good reason to be concerned about this comes from the applications of Bessel functions to partial differential equations that were sketched in the introduction to this chapter and will be discussed more fully in §5.5. The solutions of these equations involve the functions $J_\nu(\mu x)$ where μ may be very large; and the boundary conditions generally turn into equations such as $J_\nu(\mu) = 0$ or $cJ_\nu(\mu) + \mu J'_\nu(\mu) = 0$. Hence, we are particularly interested in locating the zeros of functions such as $J_\nu(x)$ or $cJ_\nu(x) + xJ'_\nu(x)$.

In this section we shall assume throughout that ν is real and x is positive. The results can, however, be extended to complex ν and x with suitable modifications.

We can obtain a clue as to the behavior of Bessel functions for large x by the following device. Suppose $f(x)$ is a solution of Bessel's equation

$$x^2 f''(x) + xf'(x) + (x^2 - \nu^2)f(x) = 0.$$

Let us set $g(x) = x^{1/2}f(x)$, so that

$$f(x) = \frac{g(x)}{x^{1/2}}, \quad f'(x) = \frac{g'(x)}{x^{1/2}} - \frac{g(x)}{2x^{3/2}}, \quad f''(x) = \frac{g''(x)}{x^{1/2}} - \frac{g'(x)}{x^{3/2}} + \frac{3g(x)}{4x^{5/2}}.$$

If we substitute these formulas into Bessel's equation and multiply through by $x^{-3/2}$, it reduces to

$$g''(x) + g(x) + \frac{\frac{1}{4} - \nu^2}{x^2} g(x) = 0.$$

Now, when x is very large the coefficient $(\frac{1}{4} - \nu^2)/x^2$ is very small, so it is reasonable to expect solutions of this equation to behave for large x like solutions of $g''(x) + g(x) = 0$. But the latter are just the linear combinations of $\sin x$ and $\cos x$ or, equivalently, functions of the form $a\cos(x + b)$ or $a\sin(x + b)$. Hence, the solutions $f(x)$ of Bessel's equation should look like $ax^{-1/2}\cos(x + b)$ or $ax^{-1/2}\sin(x + b)$.

These intuitive ideas turn out to be completely correct, and there are various ways of justifying them rigorously. It is a somewhat more arduous task to identify the particular function $ax^{-1/2}\cos(x+b)$ that corresponds to a particular solution of Bessel's equation such as $J_\nu(x)$. Nonetheless, the answer is known, and here it is.

Theorem 5.1. *For each $\nu \in \mathbf{R}$ there is a constant C_ν such that if $x \geq 1$,*

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + E_\nu(x) \quad \textit{where} \quad |E_\nu(x)| \leq \frac{C_\nu}{x^{3/2}}. \tag{5.25}$$

Thus, $J_\nu(x) \approx ax^{-1/2}\cos(x + b)$ where $a = \sqrt{2/\pi}$ and $b = (2\nu - 1)/4\pi$, with an error term that tends to zero like $x^{-3/2}$, that is, one order faster than the $x^{-1/2}$ in the main term. It is important to note that the constant C_ν in the error estimate grows with $|\nu|$. (5.25) is a useful formula for $J_\nu(x)$ only when $x \gg |\nu|$.

The proof of this result requires some rather sophisticated techniques involving Laplace transforms and contour integrals. We shall give it in §8.6. The method discussed there actually gives much more precise information about the error terms $E_\nu(x)$, and it also gives results for nonreal x and ν, but the theorem as stated here is all we shall need. Further results on the asymptotics of Bessel functions can be found in Watson [55], Chapters VII and VIII.

At this point we can explain the significance of the second solution $Y_\nu(x)$ introduced in §5.1. Replacing ν by $-\nu$ in (5.25), we have

$$J_{-\nu}(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \qquad (x \gg 0),$$

and

$$\cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right) = \cos(\nu\pi)\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \sin(\nu\pi)\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

Hence, if ν is not an integer, by combining these formulas with (5.12) and (5.25) we find that

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)} \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{5.26}$$

for $x \gg 0$, where the error term is again bounded by some constant times $x^{-3/2}$. That is, if we think of J_ν as approximately a damped cosine wave, then Y_ν is the corresponding damped sine wave. One can show that the relation (5.26) continues to hold in the limiting cases when ν is an integer.

By combining (5.25) with the recurrence formula (5.16), we also obtain an asymptotic formula for J_ν'. Indeed, we have

$$J_\nu'(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x).$$

But by (5.25),

$$\begin{aligned} J_{\nu-1}(x) &= \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{(\nu-1)\pi}{2} - \frac{\pi}{4}\right) + E_{\nu-1}(x) \\ &= -\sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + E_{\nu-1}(x), \end{aligned}$$

and

$$\left|\frac{\nu}{x} J_\nu(x)\right| \le \left(\frac{2}{\pi}\right)^{1/2} \frac{|\nu|}{x^{3/2}} + \frac{|\nu| C_\nu}{x^{5/2}}.$$

For $x \ge 1$ we have $x^{-5/2} < x^{-3/2}$, and hence

$$J_\nu'(x) = -\sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \widetilde{E}_\nu(x), \qquad |\widetilde{E}_\nu(x)| \le \frac{C_\nu'}{x^{3/2}}. \tag{5.27}$$

Of course this is the result we would get by differentiating (5.25) *if* we knew that the derivative $E_\nu'(x)$ of the error term is also dominated by $x^{-3/2}$. But since the derivative of a small function need not be small, this is not automatic; the recurrence formula saves us the trouble of calculating E_ν'.

We now turn to the problem of describing the positive solutions of the equations

$$aJ_\nu(x) + bxJ_\nu'(x) = 0 \tag{5.28}$$

where $\nu \ge 0$, $a, b \in \mathbf{R}$, and $(a, b) \ne (0, 0)$. As we have indicated, these will be of importance in solving boundary value problems.

In the first place, the function $x^{-\nu}[aJ_\nu(x) + bJ_\nu'(x)]$ is an entire analytic function of the complex variable x, so its zeros are all isolated; that is, there are only finitely many zeros in any bounded region of the complex plane. It follows that the positive solutions of the equation (5.28) can be arranged in an increasing sequence,

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$

with $\lim \lambda_j = \infty$. The main features of interest in this sequence are (i) the location of the first few terms λ_1, λ_2, etc., and (ii) the asymptotic behavior of λ_k as $k \to \infty$. (There is a sort of "grey area" in between where not much precise information is known.) To investigate the second aspect, we must distinguish between the cases $b = 0$ and $b \ne 0$; that is, we consider separately the equations

$$J_\nu(x) = 0 \qquad \text{and} \qquad cJ_\nu(x) + xJ_\nu'(x) = 0 \quad (c = a/b).$$

First, the case $b = 0$. We can read off the asymptotics of the sequence λ_k of positive zeros of J_ν immediately from the preceding results on the asymptotics of the functions J_ν and J'_ν. Indeed, from (5.25) we know that $J_\nu(x)$ is approximately $x^{-1/2}\cos(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)$ for large x, so its zeros should occur at approximately the same places as those of $\cos(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)$, namely, $(j + \frac{1}{2}\nu + \frac{3}{4})\pi$ for large positive integers j. This can be made precise by the following lemma.

Lemma 5.1. *Suppose $f(x)$ is a differentiable real-valued function that satisfies*

$$|f(x) - \cos x| \le \epsilon \quad \textit{and} \quad |f'(x) + \sin x| \le \epsilon \quad \textit{for} \quad x \ge M\pi,$$

where $\epsilon \ll 1$. Then for all integers $m \ge M$, f has exactly one zero z_m in each interval $[m\pi, (m+1)\pi]$, and it satisfies $|z_m - (m + \frac{1}{2}\pi)| < 2\epsilon$.

Proof: We shall sketch the ideas and leave it to the reader to make the details precise. First, we have $\cos m\pi = (-1)^m$ and $\cos(m + \frac{1}{2})\pi = 0$. Since $|f(x) - \cos x| < \epsilon$, f has opposite signs at $m\pi$ and $(m + 1)\pi$, so it must have at least one zero in between, and all such zeros must occur near $(m + \frac{1}{2})\pi$. But $\sin(m + \frac{1}{2}\pi) = (-1)^m$, so the condition $|f'(x) + \sin x| < \epsilon$ implies that $f'(x) \neq 0$ for x near $(m + \frac{1}{2})\pi$. Hence f is strictly increasing or decreasing near $(m + \frac{1}{2})\pi$, so it can have at most one zero there. ∎

We can apply Lemma 5.1 to the function

$$f(x) = \tilde{x}^{1/2} J_\nu(\tilde{x}), \qquad \tilde{x} = x + \tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi.$$

Since $f'(x) = \tilde{x}^{1/2} J'_\nu(\tilde{x}) + \frac{1}{2}\tilde{x}^{-1/2} J_\nu(\tilde{x})$,the estimates (5.25) and (5.27) show that $f(x)$ and $f'(x)$ differ from $\cos x$ and $-\sin x$ by errors that are bounded by a constant times x^{-1} and so can be made as small as we please by taking x large enough. The conclusion is that for large M, the solutions of the equation $J_\nu(x) = 0$ such that $x > M\pi$ are approximately at the points $(m + \frac{1}{2}\nu + \frac{3}{4})\pi$ where m is an integer. Moreover, the approximation gets better the larger we take M.

A similar argument applies to the function

$$f(x) = c\tilde{x}^{-1/2} J_\nu(\tilde{x}) + \tilde{x}^{1/2} J'_\nu(\tilde{x}), \qquad \tilde{x} = x + \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi.$$

Indeed, the second term in $f(x)$ is approximately $\sin(x + \frac{1}{2}\pi) = \cos x$ by (5.27) (note that we defined $\tilde{x}$ so as to make this shift of $\frac{1}{2}\pi$!), whereas the first term is dominated by x^{-1} by (5.25). Moreover,

$$f'(x) = -\tfrac{1}{2}c\tilde{x}^{-3/2} J_\nu(\tilde{x}) + c\tilde{x}^{-1/2} J'_\nu(\tilde{x}) + \tfrac{1}{2}\tilde{x}^{-1/2} J'_\nu(\tilde{x}) + x^{1/2} J''_\nu(\tilde{x}).$$

The first three terms are all dominated by $\tilde{x}^{-1}$, and Bessel's equation says that

$$J''_\nu(\tilde{x}) = -\tilde{x}^{-1} J'_\nu(\tilde{x}) - (1 - \nu^2\tilde{x}^{-2}) J_\nu(\tilde{x}),$$

so

$$f'(x) = \tilde{x}^{1/2} J_\nu(\tilde{x}) + (\text{terms of order } x^{-1}).$$

(5.25) then implies that $f'(x)$ is approximately $\cos(x + \frac{1}{2}\pi) = -\sin x$. Therefore, Lemma 5.1 can be applied to $f(x)$ to conclude that the function $cJ_\nu(\tilde{x}) + \tilde{x}J_\nu'(\tilde{x}) = x^{1/2} f(x)$ has zeros approximately at the points $(m + \frac{1}{2})\pi$ for large integers m. In other words, $cJ_\nu(x) + xJ_\nu'(x)$ has zeros approximately at $(m + \frac{1}{2}\nu + \frac{1}{4})\pi$ for large integers m.

There remains the question of locating the small positive zeros of $aJ_\nu(x) + bxJ_\nu'(x)$. We shall content ourselves with deriving a simple lower bound for the *smallest* positive zeros of these functions under the conditions $\nu \geq 0$ and $a, b \geq 0$. (The cases when $\nu < 0$, $a < 0$, or $b < 0$ arise only infrequently in applications.)

Lemma 5.2. *Suppose $\nu \geq 0$, $a, b \geq 0$, and $(a, b) \neq (0, 0)$. If ω_ν is the smallest positive zero of $aJ_\nu(x) + bxJ_\nu'(x)$, then $\omega_\nu > \nu$.*

Proof: The case $\nu = 0$ is trivial, so we assume $\nu > 0$. $J_\nu(x)$ and $J_\nu'(x)$ are clearly positive for small $x > 0$, since the leading terms of their power series (namely, $x^\nu/2^\nu\Gamma(\nu + 1)$ and $x^{\nu-1}/2^\nu\Gamma(\nu)$) are positive. Now, Bessel's equation can be written as

$$x\frac{d}{dx}\left[xJ_\nu'(x)\right] = (\nu^2 - x^2)J_\nu(x).$$

If the first zero ζ_ν of J_ν were $\leq \nu$, the expression on the right would be positive on the interval $(0, \zeta_\nu)$; hence $xJ_\nu'(x)$ would be increasing on this interval; hence J_ν' would be positive on this interval, which is impossible by Rolle's theorem. Therefore $\zeta_\nu > \nu$; the expression on the right *is* positive on the interval $(0, \nu)$; hence $xJ_\nu'(x)$ *is* increasing on $(0, \nu]$, so $J_\nu' > 0$ on $(0, \nu]$. We have now shown that $J_\nu(x)$ and $J_\nu'(x)$ are positive on $(0, \nu]$; but then so is $aJ_\nu(x) + bxJ_\nu'(x)$, so $\omega_\nu > \nu$. ∎

Lemma 5.2 will suffice for our purposes, but more precise estimates on these zeros are available. In particular, one can show that

$$\nu < \omega_\nu < \sqrt{2(\nu + 1)(\nu + 3)}. \tag{5.29}$$

(See Watson [55], §15.3.) Thus ω_ν is of the same order of magnitude as ν.

We sum up our results in a theorem.

Theorem 5.2. *Suppose $\nu \in \mathbf{R}$, $a, b \geq 0$, and $(a, b) \neq (0, 0)$. Let $\lambda_1, \lambda_2, \ldots$ be the positive zeros of $aJ_\nu(x) + bxJ_\nu'(x)$, arranged in increasing order. Then:*

(a) $\lambda_1 > \nu$.

(b) If $b = 0$, there is an integer $M = M(\nu)$ such that

$$\lambda_k \sim (k + M + \tfrac{1}{2}\nu + \tfrac{3}{4})\pi \quad \text{as} \quad k \to \infty.$$

(c) If $b > 0$, there is an integer $M = M(\nu, a/b)$ such that

$$\lambda_k \sim (k + M + \tfrac{1}{2}\nu + \tfrac{1}{4})\pi \quad \text{as} \quad k \to \infty.$$

Here "$\sim$" means that the difference between the quantities on the left and on the right tends to zero as $k \to \infty$.

EXERCISES

1. Fill in the details of the proof of Lemma 5.1.
2. In the text and exercises of §5.2, J_ν was computed explicitly for $\nu = \pm\frac{1}{2}, \pm\frac{3}{2}$. Verify Theorem 5.1 in these cases.
3. Use Exercise 2 and the recurrence formulas to prove Theorem 5.1 when ν is a half-integer. (Proceed by induction on n, where $\nu = \pm n + \frac{1}{2}$.)
4. Let $\{\lambda_k\}$ be the positive zeros of J_ν ($\nu \in \mathbf{R}$). Show that $J_{\nu+1}(\lambda_k) \approx \pm\sqrt{2/\pi\lambda_k}$ for large k. (In view of Theorem 5.3(a) below, this is of interest in estimating the coefficients in Fourier-Bessel series.)
5. (The interlacing theorem) Suppose $\nu \in \mathbf{R}$. Prove that between every two positive zeros of J_ν there is a zero of $J_{\nu+1}$, and between every two positive zeros of $J_{\nu+1}$ there is a zero of J_ν. (Hint: Use Rolle's theorem and the recurrence formulas (5.13) and (5.14).)
6. Let $f(x) = x^{1/2}J_\nu(x)$. Then, as shown in the text, f satisfies $f'' + f = (\nu^2 - \frac{1}{4})x^{-2}f$.
   a. Use this differential equation to show that for $n = 1, 2, 3, \ldots,$
   $$\int_{2n\pi}^{(2n+1)\pi} (\tfrac{1}{4} - \nu^2)x^{-2}f(x)\sin x\,dx = -\Big[f\big((2n+1)\pi\big) + f(2n\pi)\Big].$$
   b. Suppose $-\frac{1}{2} < \nu < \frac{1}{2}$. Show that f must vanish somewhere in the interval $[2n\pi, (2n+1)\pi]$. (Hint: By comparing signs on the two sides of the equation in part (a), show that it is impossible for f to be everywhere positive or everywhere negative on this interval.) Note that Theorem 5.1 yields a sharper result when n is large, but this elementary argument is valid for all $n \geq 1$.
7. Exercise 6 shows that J_ν has infinitely many positive zeros when $-\frac{1}{2} < \nu < \frac{1}{2}$, and the same is obviously true of $J_{1/2}(x) = \sqrt{2/\pi x}\,\sin x$. Use this fact together with Exercise 5 (but without invoking Theorem 5.1) to show that J_ν has infinitely many positive zeros for all real ν.
8. Let j_ν denote the smallest positive zero of J_ν. Show that $j_{\nu-1} < j_\nu$ for all $\nu \geq 1$. (Hint: Use formula (5.14) and Rolle's theorem.)

5.4 Orthogonal sets of Bessel functions

We recall that the differential equation (5.1) from which we derived Bessel's equation is

$$x^2 f''(x) + x f'(x) + (\mu^2 x^2 - \nu^2) f(x) = 0, \tag{5.30}$$

and that the solutions of this equation are the functions $f(x) = g(\mu x)$ where g satisfies Bessel's equation (i.e., equation (5.30) with $\mu = 1$). Upon dividing through by x, (5.30) can be rewritten as

$$x f''(x) + f'(x) - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = \big[x f'(x)\big]' - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = 0. \tag{5.31}$$

This is a Sturm-Liouville equation of the sort studied in §3.5, that is,

$$(rf')' + pf + \mu^2 wf = 0 \quad \text{where } r(x) = x,\ p(x) = -\frac{\nu^2}{x},\ w(x) = x.$$

If we consider this equation on an interval $[a, b]$ with $0 < a < b < \infty$ and impose suitable boundary conditions, say

$$\alpha f(a) + \alpha' f'(a) = 0, \qquad \beta f(b) + \beta' f'(b) = 0,$$

we obtain a regular Sturm-Liouville problem. The eigenfunctions will be of the form

$$f(x) = c_\mu J_\nu(\mu x) + d_\mu Y_\nu(\mu x) \tag{5.32}$$

where μ, c_μ, and d_μ must be chosen so that the boundary conditions hold. In this way we obtain an orthonormal basis of $L^2_w(a, b)$ consisting of functions of the form (5.32), where $w(x) = x$. In general the determination of the eigenvalues μ^2 and the coefficients c_μ and d_μ is a rather messy business, and we shall not pursue the matter further.

More important and more interesting, however, is to consider the equation (5.31) on an interval $[0, b]$ under the assumption $\nu \geq 0$. Here the Sturm-Liouville problem is singular, because the leading coefficient r vanishes at $x = 0$ and the coefficient p blows up there. As a result, it is inappropriate to impose the usual sort of boundary condition at $x = 0$ such as $\alpha f(0) + \alpha' f'(0) = 0$. Indeed, we know that the solutions are of the form (5.32), and such functions (and their derivatives) become infinite at $x = 0$ unless $d_\mu = 0$. Instead, the natural boundary condition at $x = 0$ is simply that the solution should be continuous there, i.e., that $d_\mu = 0$. We can still impose a boundary condition at $x = b$, so the Sturm-Liouville problem we propose to consider is

$$\begin{gathered} xf''(x) + f'(x) - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = 0 \qquad (\nu \geq 0), \\ f(0+) \text{ exists and is finite}, \qquad \beta f(b) + \beta' f'(b) = 0. \end{gathered} \tag{5.33}$$

The results of §3.5 concerning the reality of the eigenvalues and the orthogonality of the eigenfunctions for regular Sturm-Liouville problems are still valid in the present situation. What needs to be checked is that if f and g are eigenfunctions of (5.33), that is,

$$f(x) = J_\nu(\mu_j x), \qquad g(x) = J_\nu(\mu_k x),$$

then

$$\langle L(f), g \rangle = \langle f, L(g) \rangle \quad \text{where } L(f) = (xf')' - \frac{\nu^2}{x} f. \tag{5.34}$$

However, if we apply Lagrange's identity to this operator L on the interval $[\epsilon, b]$, we find that

$$\int_\epsilon^b \left[L(f)(x)\overline{g(x)} - f(x)\overline{L(g)(x)} \right] dx = \left[xf'(x)\overline{g(x)} - xf(x)\overline{g'(x)} \right]_\epsilon^b.$$

The boundary condition at $x = b$ causes the endpoint evaluation at $x = b$ to vanish. As for $x = \epsilon$: If $\nu > 0$, then $f(x)$, $g(x)$, $xf'(x)$, and $xg'(x)$ are asymptotic to constant multiples of x^ν as $x \to 0$, so

$$|\epsilon f'(\epsilon)\overline{g(\epsilon)} - \epsilon f(\epsilon)\overline{g'(\epsilon)}| \leq C\epsilon^{2\nu} \to 0 \quad \text{as } \epsilon \to 0.$$

If $\nu = 0$, then $f(0) = g(0) = 1$ and $f'(0) = g'(0) = 0$, so again the contribution at $x = \epsilon$ vanishes as $\epsilon \to 0$. In either case, we have verified (5.34).

Once this is known, the proof of Theorem 3.9 in §3.5 goes through to show that the eigenvalues of (5.33) are real, the eigenfunctions are orthogonal with respect to the weight function $w(x) = x$, and the eigenspaces are 1-dimensional.

Now, if $f(x) = J_\nu(\mu x)$, then $f'(x) = \mu J_\nu'(\mu x)$. Hence, the solutions of (5.33) are the functions $J_\nu(\mu x)$ such that

$$\beta J_\nu(\mu b) + \beta' \mu J_\nu'(\mu b) = 0.$$

It will now be convenient to set $\lambda = \mu b$, so that $\mu = \lambda / b$. We distinguish between $\beta' = 0$, in which case we have

$$J_\nu(\lambda) = 0, \tag{5.35}$$

and $\beta' \neq 0$, in which case we set $c = b\beta/\beta'$ and obtain

$$cJ_\nu(\lambda) + \lambda J_\nu'(\lambda) = 0. \tag{5.36}$$

Equations (5.35) and (5.36) are of the sort we analyzed in §5.3. In either case there is an infinite sequence $\{\lambda_k\}_1^\infty$ of positive solutions, and the corresponding eigenvalues of problem (5.33) are the numbers λ_k^2/b^2.

Thus we have identified the positive eigenvalues of the problem (5.33). There remains the question of zero or negative eigenvalues, concerning which we have the following result.

Lemma 5.3. *Zero is an eigenvalue of (5.33) if and only if $\beta/\beta' = -\nu/b$, in which case the eigenfunction is $f(x) = x^\nu$. If $\beta' = 0$ or if $\beta/\beta' \geq -\nu/b$, there are no negative eigenvalues.*

Proof: When $\mu = 0$, the differential equation in (5.33) becomes the Euler equation

$$x^2 f''(x) + xf'(x) - \nu^2 f(x) = 0,$$

which we analyzed in §4.3. The general solution is $c_1 x^\nu + c_2 x^{-\nu}$ if $\nu > 0$, or $c_1 + c_2 \log x$ if $\nu = 0$. The boundary condition at $x = 0$ forces $c_2 = 0$, and then the boundary condition at $x = b$ becomes $\beta b + \nu\beta' = 0$. This proves the first assertion.

To investigate negative eigenvalues, i.e., the case $\mu^2 < 0$ in (5.33), we set $\mu = i\kappa$. The general solution of (5.33) is then $c_1 J_\nu(i\kappa x) + c_2 Y_\nu(i\kappa x)$, and again the boundary condition at $x = 0$ forces $c_2 = 0$. (The behavior of $Y_\nu(x)$ that we described in §5.1 still holds when x is imaginary; in particular, $Y_\nu(x)$ blows up as $x \to 0$.) Hence the boundary condition at $x = b$ is still (5.35) or (5.36), with $\lambda = i\kappa b$, so we must investigate solutions of the equations

$$J_\nu(iy) = 0 \quad \text{or} \quad cJ_\nu(iy) + iyJ'_\nu(iy) = 0, \quad (c = b\beta/\beta',\ y > 0). \tag{5.37}$$

Now, from the defining formula (5.10) for J_ν we have

$$J_\nu(iy) = i^\nu I_\nu(y), \quad \text{where } I_\nu(y) = \sum_0^\infty \frac{1}{k!\Gamma(\nu + k + 1)} \left(\frac{y}{2}\right)^{\nu+2k}.$$

Moreover, the recurrence formula (5.15), which is valid for all complex x, shows that

$$\begin{aligned} iyJ'_\nu(iy) &= \nu J_\nu(iy) - iyJ_{\nu+1}(iy) = \nu i^\nu I_\nu(y) - i^{\nu+2} y I_{\nu+1}(y) \\ &= i^\nu \left[\nu I_\nu(y) + yI_{\nu+1}(y)\right], \end{aligned}$$

so (5.37) can be written as

$$I_\nu(y) = 0 \quad \text{or} \quad (c + \nu)I_\nu(y) + yI_{\nu+1}(y) = 0 \qquad (y > 0).$$

But it is obvious from the above definition of I_ν that $I_\nu(y) > 0$ (and likewise $I_{\nu+1}(y) > 0$) for all $y \neq 0$. Hence the first equation has no solutions, and neither does the second one when $c + \nu = (b\beta/\beta') + \nu \geq 0$. This completes the proof.

(A slightly more careful analysis would show that when $\beta/\beta' < -\nu/b$, there is exactly one negative eigenvalue; the situation is similar to the one analyzed in the examples of §3.5. However, this case arises only rarely in applications.) ∎

We have therefore constructed a whole family (depending on the parameters ν, β, and β') of orthogonal sets of functions on $[0, b]$, with respect to the weight $w(x) = x$, of the form $f_k(x) = J_\nu(\lambda_k x/b)$. In order to make this a useful tool for solving concrete problems, we need also to identify the norms of these functions, namely,

$$\|f_k\|_w^2 = \int_0^b |f_k(x)|^2 x\, dx.$$

Actually, all these eigenfunctions are real, so we can omit the absolute values here, and we have the following result.

Lemma 5.4. *If $\mu > 0$, $b > 0$, and $\nu \geq 0$,*

$$\int_0^b J_\nu(\mu x)^2 x\, dx = \frac{b^2}{2} J'_\nu(\mu b)^2 + \frac{\mu^2 b^2 - \nu^2}{2\mu^2} J_\nu(\mu b)^2. \tag{5.38}$$

Proof: Let $f(x) = J_\nu(\mu x)$. The differential equation satisfied by f is

$$x^2 f'' + xf' + (\mu^2 x^2 - \nu^2)f = 0, \quad \text{or} \quad x(xf')' = (\nu^2 - \mu^2 x^2)f.$$

If we multiply this through by $2f'$, it becomes

$$2(xf')'(xf') = (\nu^2 - \mu^2 x^2)(2f'f), \quad \text{or} \quad \left[(xf')^2\right]' = (\nu^2 - \mu^2 x^2)(f^2)'.$$

We integrate from 0 to b and use integration by parts on the right side:

$$(xf')^2|_0^b = (\nu^2 - \mu^2 x^2)f^2|_0^b + \mu^2 \int_0^b f(x)^2(2x)\,dx.$$

The endpoint evaluations at $x = 0$ all vanish. Indeed, this is obvious for $(xf')^2$ and $\mu^2 x^2 f^2$; and it is true for $\nu^2 f^2$ because either $\nu = 0$ or $\nu > 0$, and in the latter case $f(0) = J_\nu(0) = 0$. Therefore, we have

$$2\mu^2 \int_0^b f(x)^2 x\,dx = b^2 f'(b)^2 + (\mu^2 b^2 - \nu^2)f(b)^2,$$

and since $f'(x) = \mu J_\nu'(\mu x)$, this proves (5.38). ∎

Equation (5.38) can be simplified in the cases where $\mu = \lambda/b$ and the boundary condition (5.35) or (5.36) is satisfied, namely,

$$\text{if } J_\nu(\lambda) = 0, \quad \int_0^b J_\nu\left(\frac{\lambda x}{b}\right)^2 x\,dx = \frac{b^2}{2} J_\nu'(\lambda)^2, \tag{5.39}$$

$$\text{if } cJ_\nu(\lambda) + \lambda J_\nu'(\lambda) = 0, \quad \int_0^b J_\nu\left(\frac{\lambda x}{b}\right)^2 x\,dx = \frac{b^2(\lambda^2 - \nu^2 + c^2)}{2\lambda^2} J_\nu(\lambda)^2. \tag{5.40}$$

It is customary to restate (5.39) by using the recurrence formula (5.15), which implies that $J_\nu'(\lambda) = -J_{\nu+1}(\lambda)$ whenever $J_\nu(\lambda) = 0$, thus:

$$\text{if } J_\nu(\lambda) = 0, \quad \int_0^b J_\nu\left(\frac{\lambda x}{b}\right)^2 x\,dx = \frac{b^2}{2} J_{\nu+1}(\lambda)^2. \tag{5.41}$$

We can now write down a number of orthonormal sets of Bessel functions on $[0, b]$ obtained from the Sturm-Liouville problems (5.33). One last, crucial question remains: *Are these sets orthonormal bases?* The theorems of §3.5 do not apply since these Sturm-Liouville problems are singular. Nonetheless, the answer is *yes*, and we shall explain a method for proving this in §10.4; a complete proof can be found in Watson [55], Chapter XVIII. We sum up the results in a theorem.

Theorem 5.3. *Suppose $\nu \ge 0$, $b > 0$, and $w(x) = x$.*

(a) Let $\{\lambda_k\}_1^\infty$ be the positive zeros of $J_\nu(x)$, and let $\phi_k(x) = J_\nu(\lambda_k x/b)$. Then $\{\phi_k\}_1^\infty$ is an orthogonal basis for $L_w^2(0, b)$, and

$$\|\phi_k\|_w^2 = \frac{b^2}{2} J_{\nu+1}(\lambda_k)^2.$$

(b) Suppose $c \ge -\nu$. Let $\{\tilde\lambda_k\}_1^\infty$ be the positive zeros of $cJ_\nu(x) + xJ_\nu'(x)$, and let $\psi_k(x) = J_\nu(\tilde\lambda_k x/b)$. If $c > -\nu$, then $\{\psi_k\}_1^\infty$ is an orthogonal basis for $L_w^2(0, b)$. If $c = -\nu$, then $\{\psi_k\}_0^\infty$ is an orthogonal basis for $L_w^2(0, b)$, where $\psi_0(x) = x^\nu$. Moreover,

$$\|\psi_k\|_w^2 = \frac{b^2(\tilde\lambda_k^2 - \nu^2 + c^2)}{2\tilde\lambda_k^2} J_\nu(\tilde\lambda_k)^2 \quad (k \ge 1), \qquad \|\psi_0\|_w^2 = \frac{b^{2\nu+2}}{2\nu+2}.$$

In practice, the constant c in part (b) is almost always nonnegative. Under this condition the Bessel functions $\{\psi_k\}_1^\infty$ form an orthogonal basis for $L_w^2(0,b)$ except in the single case $c = \nu = 0$, when one must add in the constant function $\psi_0(x) = 1$. The latter case is an important one, however, and one must not forget its exceptional character.

From Theorem 5.3 we know that any $f \in L_w^2(0,b)$ can be expanded in a **Fourier-Bessel series**

$$f = \sum c_k \phi_k, \qquad c_k = \frac{1}{\|\phi_k\|_w^2} \int_0^b f(x)\phi_k(x)x\,dx,$$

$$\text{or} \quad f = \sum d_k \psi_k, \qquad d_k = \frac{1}{\|\psi_k\|_w^2} \int_0^b f(x)\psi_k(x)x\,dx.$$

(The second of these expansions is also called a **Dini series**.) These series converge in the norm of $L_w^2(0,b)$, but under suitable conditions one can also prove pointwise or uniform convergence. In fact, except perhaps at the endpoints 0 and b, the behavior of these series is much like ordinary Fourier series. For example, if f is piecewise smooth on $[0,b]$ then $\sum c_j\phi_j(x)$ and $\sum d_j\psi_j(x)$ converge to $\frac{1}{2}\left[f(x-) + f(x+)\right]$ for all $x \in (0,b)$. Of course, $\phi_j(b) = 0$ for all j because of the boundary condition, and $\phi_j(x)$ and $\psi_j(x)$ vanish to order ν as $x \to 0$ for all j; thus one cannot expect the series to converge well near the endpoints unless f satisfies similar conditions. However, if f does satisfy such conditions and is suitably smooth, one can prove absolute and uniform convergence. See Watson [55], Chapter XVIII.

Example. Let $\{\lambda_k\}$ be the positive zeros of $J_0(x)$, and let $f(x) = 1$ for $0 \le x \le b$. According to Theorem 5.3(a), we have $f(x) = \sum_1^\infty c_k J_0(\lambda_k x/b)$ (the series converging at least in the norm of $L_w^2(0,b)$ with $w(x) = x$), where

$$c_k = \frac{2}{b^2 J_1(\lambda_k)^2} \int_0^b J_0\left(\frac{\lambda_k x}{b}\right) x\,dx.$$

Since $xJ_0(x)$ is the derivative of $xJ_1(x)$ by the recurrence formula (5.14), we make the substitution $x = bt/\lambda_k$ and obtain

$$c_k = \frac{2}{b^2 J_1(\lambda_k)^2} \frac{b^2}{\lambda_k^2} \int_0^{\lambda_k} J_0(t)\,t\,dt = \frac{2}{\lambda_k^2 J_1(\lambda_k^2)} \left[tJ_1(t)\right]_0^{\lambda_k} = \frac{2}{\lambda_k J_1(\lambda_k)}.$$

Other examples will be found in the exercises; the evaluations of the integrals are usually applications of the recurrence formulas.

EXERCISES

In Exercises 1–4, expand the given function on the interval $[0,b]$ in a Fourier-Bessel series $\sum c_k J_0(\lambda_k x/b)$ where $\{\lambda_k\}_1^\infty$ are the positive zeros of J_0.

1. $f(x) = x^2$. (Hint: Exercise 8, §5.2.)
2. $f(x) = b^2 - x^2$. (Hint: Exercise 1.)
3. $f(x) = x$. (Hint: Exercise 8, §5.2.)
4. $f(x) = 1$ for $0 \le x \le \frac{1}{2}b$, $f(x) = 0$ for $\frac{1}{2}b < x \le b$.
5. Expand $f(x) = 1$ on the interval $[0, b]$ in a series $\sum c_k J_0(\lambda_k x/b)$ where $\{\lambda_k\}$ are the positive zeros of $cJ_0(x) + xJ_0'(x)$, $c > 0$. What about the case $c = 0$? (Be careful!)
6. Expand $f(x) = x$ on the interval $[0, 1]$ in a series $\sum c_k J_1(\lambda_k x)$ where $\{\lambda_k\}$ are the positive zeros of J_1.
7. Expand $f(x) = x^\nu$ on the interval $[0, 1]$ in a series $\sum c_k J_\nu(\lambda_k x)$ where $\nu > 0$ and $\{\lambda_k\}$ are the positive zeros of J_ν'.
8. Let $f(x) = x$ for $0 \le x < 1$, $f(x) = 0$ for $1 \le x \le 2$. Expand $f(x)$ on the interval $[0, 2]$ in a series $\sum c_k J_1(\lambda_k x/2)$ where $\{\lambda_k\}$ are the positive zeros of J_1'.
9. Let $\{\lambda_k\}$ be the positive zeros of J_0, and let $\phi_k(x) = J_0(\lambda_k\sqrt{x/l})$. Show that $\{\phi_k\}$ is an orthogonal basis for $L^2(0, l)$ (with weight function 1). What is the norm of ϕ_k in $L^2(0, l)$?

5.5 Applications of Bessel functions

In the introduction to this chapter we showed that if one applies separation of variables to the two-dimensional wave equation in polar coordinates,

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}),$$

one obtains the ordinary differential equations

$$\begin{gathered} T''(t) + c^2\mu^2 T(t) = 0, \qquad \Theta''(\theta) + \nu^2\Theta(\theta) = 0, \\ r^2R''(r) + rR'(r) + (\mu^2 r^2 - \nu^2)R(r) = 0. \end{gathered} \tag{5.42}$$

The heat equation works similarly, except that the equation for T is $T'(t) + k\mu^2 T(t) = 0$. Let us sketch the procedure for solving these equations.

Suppose we are interested in solving the partial differential equation in the disc of radius b about the origin, with some boundary conditions at $r = b$. In the first place, by the nature of polar coordinates, Θ must be 2π-periodic. Hence ν must be an integer n, which we can take to be nonnegative, and $\Theta(\theta) = c\cos n\theta + d\sin n\theta$. The equation for R is now the Bessel equation of order n. Since we want a solution of the partial differential equation in the whole disc including the origin, we forbid $R(r)$ to blow up at $r = 0$; hence $R(r)$ must be a constant multiple of $J_n(\mu r)$. Moreover, there will be a sequence of positive numbers μ_k for which $J_n(\mu_k r)$ satisfies the boundary conditions at $r = b$. Finally, we plug these numbers into the equation for T and solve it.

The problem of finding a solution with given initial conditions now reduces to the problem of expanding the initial data in a series involving the functions Θ and R derived above, that is, a series of the form

$$\sum_{n,k}(c_{nk}\cos n\theta + d_{nk}\sin n\theta)J_n(\mu_k r). \tag{5.43}$$

This is a *doubly infinite* series, involving the two indices n and k. However, we point out that if the θ-dependence of the initial data involves only finitely many of the functions $\cos n\theta$ and $\sin n\theta$, then only the terms in (5.43) involving those particular functions will be nonzero, so (5.43) will reduce to a finite sum of singly infinite series. For example, if the initial data are radial, i.e., independent of θ, then only the terms with $n = 0$ survive, and (5.43) reduces to $\sum a_k J_0(\mu_k r)$.

Similar remarks apply to boundary value problems in "polar-coordinate rectangles" bounded by circles $r = a$, $r = b$ and rays $\theta = \alpha$, $\theta = \beta$, except that the indices ν and eigenfunctions Θ will generally be different.

We now turn to some specific applications, in which these ideas will be explained more fully.

Vibrations of a circular membrane

Let us now solve the problem of the vibrations of a circular membrane fixed along its boundary, such as a drum, that occupies the disc of radius b centered at the origin. According to the preceding discussion, we need to solve (5.42) subject to the conditions that Θ should be 2π-periodic, that R should be continuous at $r = 0$, and that $R(b) = 0$, and we obtain

$$\Theta(\theta) = c_n\cos n\theta + d_n\sin n\theta,$$

$$R(r) = J_n\left(\frac{\lambda r}{b}\right) \quad \text{where } J_n(\lambda) = 0.$$

In particular $\mu = \lambda/b$ in (5.42), so

$$T(t) = a_1\cos\frac{\lambda ct}{b} + a_2\sin\frac{\lambda ct}{b},$$

where again λ satisfies $J_n(\lambda) = 0$.

Let $\lambda_{1,n}, \lambda_{2,n}, \lambda_{3,n}, \ldots$ be the positive zeros of $J_n(x)$. By Theorem 5.3 we know that $\{J_n(\lambda_{k,n}r/b)\}_{k=1}^{\infty}$ is an orthogonal basis for $L^2_w(0,b)$ where $w(r) = r$. Moreover, $\{\cos n\theta\}_0^\infty \cup \{\sin n\theta\}_1^\infty$ is an orthogonal basis for $L^2(-\pi,\pi)$. It follows that the products $J_n(\lambda_{k,n}r/b)\cos n\theta$ and $J_n(\lambda_{k,n}r/b)\sin n\theta$ will form an orthogonal set in $L^2_w(D)$, where

$$D = \left\{(r,\theta) : 0 \le r \le b,\ -\pi \le \theta \le \pi\right\}, \qquad w(r,\theta) = r.$$

But if we interpret r and θ as polar coordinates in the xy-plane, D is nothing but the disc of radius b about the origin, and the weighted measure $w(r,\theta)\,dr\,d\theta$ is Euclidean area measure:

$$w(r,\theta)\,dr\,d\theta = r\,dr\,d\theta = dx\,dy.$$

In fact, we have the following result.

Theorem 5.4. *Let $\lambda_{1,n}, \lambda_{2,n}, \lambda_{3,n}, \ldots$ be the positive zeros of $J_n(x)$. Then*

$$\left\{J_n\left(\frac{\lambda_{k,n}r}{b}\right)\cos n\theta : n \geq 0,\ k \geq 1\right\} \cup \left\{J_n\left(\frac{\lambda_{k,n}r}{b}\right)\sin n\theta : n, k \geq 1\right\}$$

is an orthogonal basis for $L^2(D)$, where D is the disc of radius b about the origin.

Proof: This is not an instance of Theorem 4.1 of §4.4 because we are using a different basis for functions of r for each choice of the index n; nonetheless, the argument we used to prove that theorem also proves this one. That is, one checks orthogonality by evaluating the double integrals that define the inner products as iterated integrals. To prove completeness, suppose that $f \in L^2(D)$ is orthogonal to all the functions $J_n(\lambda_{k,n}r/b)\cos n\theta$ and $J_n(\lambda_{k,n}r/b)\sin n\theta$. Then the functions

$$g_n(r) = \int_{-\pi}^{\pi} f(r,\theta)\cos n\theta\, d\theta \quad \text{and} \quad h_n(r) = \int_{-\pi}^{\pi} f(r,\theta)\sin n\theta\, d\theta$$

are orthogonal to all the functions $J_n(\lambda_{k,n}r/b)$ $(k = 1, 2, \ldots)$ and hence are zero. But this says that for (almost) every r, $f(r,\theta)$ is orthogonal to all the functions $\cos n\theta$ and $\sin n\theta$; hence $f = 0$. ∎

Now we can solve the vibrating membrane problem with initial conditions. For simplicity, let us take them to be

$$u(r,\theta,0) = f(r,\theta), \qquad u_t(r,\theta,0) = 0.$$

The initial condition $u_t = 0$ means that we must drop the sine term in $T(t)$. Hence, taking u to be a general superposition of the solutions we have constructed,

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(c_{nk}\cos n\theta + d_{nk}\sin n\theta)J_n\left(\frac{\lambda_{k,n}r}{b}\right)\cos\frac{\lambda_{k,n}ct}{b}. \tag{5.44}$$

To solve the problem we have merely to determine the coefficients c_{nk} and d_{nk} so that $u(r,\theta,0) = f(r,\theta)$, and this means expanding f in terms of the basis of Theorem 5.4. In view of the normalizations for the Bessel functions presented in Theorem 5.3, we have

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(c_{nk}\cos n\theta + d_{nk}\sin n\theta)J_n\left(\frac{\lambda_{k,n}r}{b}\right),$$

where

$$c_{0k} = \frac{1}{\pi b^2 J_1(\lambda_{k,0})^2}\int_{-\pi}^{\pi}\int_0^b f(r,\theta)J_0\left(\frac{\lambda_{k,0}r}{b}\right)r\,dr\,d\theta,$$

and for $n \geq 1$,

$$c_{nk} = \frac{2}{\pi b^2 J_{n+1}(\lambda_{k,n})^2}\int_{-\pi}^{\pi}\int_0^b f(r,\theta)J_n\left(\frac{\lambda_{k,n}r}{b}\right)\cos(n\theta)r\,dr\,d\theta,$$

$$d_{nk} = \frac{2}{\pi b^2 J_{n+1}(\lambda_{k,n})^2}\int_{-\pi}^{\pi}\int_0^b f(r,\theta)J_n\left(\frac{\lambda_{k,n}r}{b}\right)\sin(n\theta)r\,dr\,d\theta.$$

Example 1. Suppose the initial displacement is $f(r, \theta) = b^2 - r^2$. Since f is independent of θ, so is u; hence the only nonzero terms in (5.44) are the ones with $n = 0$. That is,

$$u(r,t) = \sum_1^\infty c_k J_0\left(\frac{\lambda_k r}{b}\right) \cos\frac{\lambda_k ct}{b} \qquad (\lambda_k = \lambda_{k,0}),$$

where $\sum c_k J_0(\lambda_k r/b)$ is the Fourier-Bessel expansion of the function $b^2 - r^2$. Therefore, by Exercise 2 of §5.4,

$$u(r,t) = \sum_1^\infty \frac{8b^2}{\lambda_k^3 J_1(\lambda_k)} J_0\left(\frac{\lambda_k r}{b}\right) \cos\frac{\lambda_k ct}{b}.$$

The most interesting aspect of the solution (5.44) is the set of allowable frequencies, namely, the zeros of the Bessel functions,

$$\left\{\frac{\pi c\lambda_{k,n}}{b} : n \geq 0,\ k \geq 1\right\}.$$

An important feature of this set is that there are only finitely many frequencies less than a preassigned number M. Indeed, from Theorem 5.3 we know that all zeros of $J_n(\lambda)$ satisfy $\lambda > n$, i.e., $\lambda_{k,n} > n$ for all k and n. Consequently, if we are to have $\pi c\lambda_{k,n}/b < M$, we must have $n < bM/\pi c$. That is, for each n J_n has only finitely many zeros $\lambda_{k,n}$ satisfying $\pi c\lambda_{k,n}/b < M$, and if $n \geq bM/\pi c$ it has none at all; hence there are only finitely many altogether.

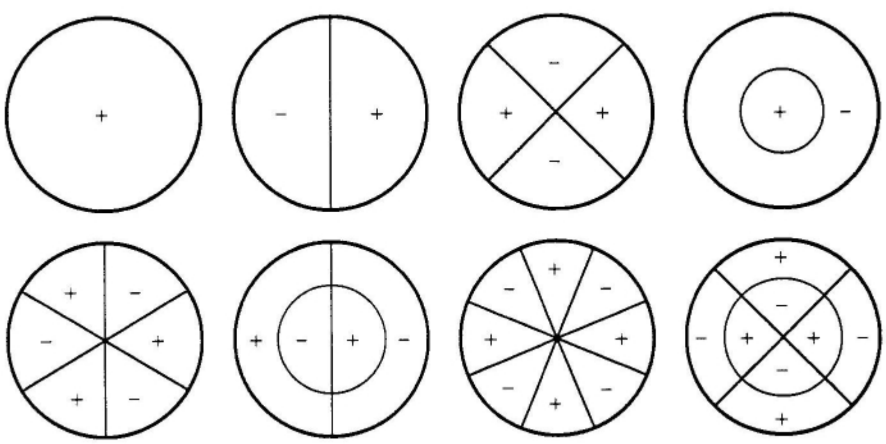


FIGURE 5.2. Diagrams of the eigenfunctions $J_n(\lambda_{k,n} r)\cos n\theta$ of the vibrating membrane for the eight smallest $\lambda_{k,n}$'s. The plus or minus signs indicate the regions where the eigenfunctions are positive or negative. Top: $\lambda_{1,0}$, $\lambda_{1,1}$, $\lambda_{1,2}$, and $\lambda_{2,0}$. Bottom: $\lambda_{1,3}$, $\lambda_{2,1}$, $\lambda_{1,4}$, and $\lambda_{2,2}$.

The smallest numbers $\lambda_{k,n}$ are as follows, correct to two decimal places.

$$\begin{array}{lllll} \lambda_{1,0} = 2.40, & \lambda_{1,1} = 3.83, & \lambda_{1,2} = 5.14, & \lambda_{2,0} = 5.52, & \lambda_{1,3} = 6.38, \\ \lambda_{2,1} = 7.02, & \lambda_{1,4} = 7.59, & \lambda_{2,2} = 8.42, & \lambda_{3,0} = 8.65, & \lambda_{1,5} = 8.77. \end{array}$$

The eigenfunctions corresponding to the first eight of these are drawn schematically in Figure 5.2. Observe that the frequencies are not integer multiples of a fundamental frequency, even approximately; hence drums have poorer tone quality than strings or wind instruments. In practice, drums that are designed to have a definite pitch possess structural features that make our simple mathematical model rather inaccurate. (For example, the vibrating membranes of Indian drums such as the tabla or mridangam are of nonuniform thickness.) It should also be mentioned that the pitch of a drum depends on how the drum is struck. If it is struck at the center, only the frequencies $\pi c\lambda_{k,0}/b$ (corresponding to the circularly symmetric vibrations with $n = 0$ in (5.44)) are significant. But if it is struck near the edge, as kettledrums normally are, the predominant frequency is likely to be $\pi c\lambda_{1,1}/b$. See Rossing [46] for a discussion of the physics of kettledrums.

The heat equation in polar coordinates

The heat equation in polar coordinates is

$$u_t = K(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}).$$

(We call the diffusivity coefficient K, since we are using k as an index for the zeros of Bessel functions.) We may imagine a solid body occupying some region in cylindrical coordinates,

$$R = \left\{(r, \theta, z) : (r, \theta) \in D,\ z_1 \leq z \leq z_2\right\}$$

(D being a region in the plane), in which the temperature, for one reason or another, is independent of z. Suppose D is the disc of radius b about the origin. If we impose the condition that $u = 0$ on the boundary of D, then exactly the same analysis as before leads to a solution that looks like (5.44) except that $\cos \lambda_{k,n}ct/b$ is replaced by $\exp[-\lambda_{k,n}^2 Kt/b^2]$. If instead we impose the "Newton's law of cooling" boundary condition $u_r + cu = 0$ $(c > 0)$, the results are similar except that the numbers $\lambda_{k,n}$ should be the positive zeros of $bcJ_n(x) + xJ_n'(x)$.

Rather than work out these problems in detail, let us do a problem with some new features. Suppose D is the wedge-shaped region

$$D = \left\{(r, \theta) : 0 \leq r \leq b,\ 0 \leq \theta \leq \alpha\right\},$$

where $0 < \alpha < 2\pi$, and let us suppose that the boundary is insulated. This means that the normal derivative of u on the boundary must vanish, that is,

$$u_\theta(r, 0) = u_\theta(r, \alpha) = u_r(b, \theta) = 0.$$

If we take $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, then, separation of variables leads to the following 1-dimensional problems:

$$r^2R''(r) + rR'(r) + (\mu^2r^2 - \nu^2)R(r) = 0, \qquad R'(b) = 0;$$
$$\Theta''(\theta) + \nu^2\Theta(\theta) = 0, \qquad \Theta'(0) = \Theta'(\alpha) = 0;$$
$$T'(t) + \mu^2KT(t) = 0.$$

(There is also an implied boundary condition at $r = 0$, namely, that R should not blow up there.) The differential equation for Θ together with the boundary condition at 0 imply that (up to a constant factor) $\Theta(\theta) = \cos\nu\theta$, and the boundary condition at α then forces $\nu = n\pi/\alpha$. In short, we obtain Fourier cosine series in θ, Bessel functions of order $n\pi/\alpha$ in r, and, of course, exponential functions in t. More precisely, let $\{\lambda_{k,n}\}$ now denote the positive zeros of $J'_{n\pi/\alpha}(x)$. Then $\{J_{n\pi/\alpha}(\lambda_{k,n}r/b)\}_{k=1}^{\infty}$ is a set of eigenfunctions for the Sturm-Liouville problem in r with eigenvalues $\mu^2 = (\lambda_{k,n}/b)^2$. Also, by Theorem 5.3 it is an orthogonal basis for $L^2_w(0,b)$ with $w(r) = r$, *except* in the case $n = 0$. For $n = 0$ one must augment this set by including the constant function 1 (corresponding to the eigenvalue $\mu = 0$) in order to make it complete; and constant functions are solutions of our boundary value problem. We therefore arrive at the following general solution:

$$u(r,\theta,t) = a_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_{nk}J_{n\pi/\alpha}\left(\frac{\lambda_{k,n}r}{b}\right)\cos\left(\frac{n\pi\theta}{\alpha}\right)\exp\left(-\frac{\lambda_{k,n}^2Kt}{b^2}\right).$$

If we wish to satisfy an initial condition $u(r,\theta,0) = f(r,\theta)$, we must have

$$f(r,\theta) = a_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_{nk}J_{n\pi/\alpha}\left(\frac{\lambda_{k,n}r}{b}\right)\cos\left(\frac{n\pi\theta}{\alpha}\right). \tag{5.45}$$

But the obvious analogue of Theorem 5.4 holds here, so such an expansion is possible. In fact, taking account of the normalizations in Theorem 5.3, we find that

$$a_{00} = \frac{2}{\alpha b^2}\int_0^b\int_0^\alpha f(r,\theta)r\,dr\,d\theta,$$
$$a_{0k} = \frac{2\lambda_{k,0}^2}{\alpha b^2(\lambda_{k,0})^2J_0(\lambda_{k,0})^2}\int_0^b\int_0^\alpha f(r,\theta)J_0\left(\frac{\lambda_{k,0}r}{b}\right)r\,dr\,d\theta \qquad (k \geq 1),$$

and for $n,k \geq 1$,

$$a_{nk} = \frac{4\lambda_{k,n}^2}{\alpha b^2\left[\lambda_{k,n}^2 - (n\pi/\alpha)^2\right]J_{n\pi/\alpha}(\lambda_{k,n})^2} \times \int_0^b\int_0^\alpha f(r,\theta)J_{n\pi/\alpha}\left(\frac{\lambda_{k,n}r}{b}\right)\cos\frac{n\pi\theta}{\alpha}r\,dr\,d\theta.$$

Example 2. Suppose that $\alpha = \frac{1}{2}\pi$ and the initial temperature is $f(r,\theta) = r^2\cos 2\theta$. Then (5.45) becomes

$$r^2\cos 2\theta = a_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_{nk} J_{2n}\left(\frac{\lambda_{k,n} r}{b}\right)\cos 2n\theta.$$

But the left side involves $\cos 2n\theta$ only for $n = 1$; so it is clear by inspection, or from the orthogonality relations for $\cos 2n\theta$ on $[0, \frac{1}{2}\pi]$, that only the terms with $n = 1$ on the right will be nonzero. Hence, after canceling the $\cos 2\theta$ on both sides, we are reduced to finding the coefficients $a_k = a_{1k}$ in the expansion

$$r^2 = \sum_{k=1}^{\infty} a_k J_2\left(\frac{\lambda_k r}{b}\right), \qquad \lambda_k = \lambda_{k,1}.$$

By Theorem 5.3, these are given by

$$a_k = \frac{2\lambda_k^2}{b^2(\lambda_k^2 - 4)J_2(\lambda_k)^2}\int_0^b r^3 J_2\left(\frac{\lambda_k r}{b}\right) dr,$$

and by the recurrence formula (5.14),

$$\int_0^b r^3 J_2\left(\frac{\lambda_k r}{b}\right) dr = \left(\frac{b}{\lambda_k}\right)^4 \int_0^{\lambda_k} x^3 J_2(x)\,dx = \left(\frac{b}{\lambda_k}\right)^4 \lambda_k^3 J_3(\lambda_k).$$

Combining these results, we obtain the solution:

$$u(r,\theta,t) = 2b^2\cos 2\theta \sum_1^{\infty} \frac{\lambda_k J_3(\lambda_k)}{(\lambda_k^2 - 4)J_2(\lambda_k)^2} J_2\left(\frac{\lambda_k r}{b}\right)\exp\left(-\frac{\lambda_k^2 K t}{b^2}\right).$$

We can also solve the heat equation in an annulus $0 < a \le r \le b$ with boundary conditions at $r = a$ and $r = b$. Here the eigenfunctions in θ are linear combinations of $\cos n\theta$ and $\sin n\theta$ with n an integer, just as in the disc. But in the variable r we obtain a regular Sturm-Liouville problem on the interval $[a, b]$ involving the Bessel equation of order n, and the eigenfunctions will be linear combinations of $J_n(\lambda r/b)$ and $Y_n(\lambda r/b)$ chosen so as to satisfy the boundary conditions. See Exercise 8.

The Dirichlet problem in a cylinder

As a final application, let us consider the Dirichlet problem in the cylinder

$$D = \{(r,\theta,z) : 0 \le r \le b,\ 0 \le z \le l\}.$$

That is, we wish to solve

$$\begin{aligned} &u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \quad \text{in } D, \\ &u(r,\theta,0) = f(r,\theta), \qquad u(r,\theta,l) = g(r,\theta), \qquad u(b,\theta,z) = h(\theta,z). \end{aligned} \tag{5.46}$$

Here we shall work out the special case where $f = h = 0$ and g is independent of θ:

$$u(r,\theta,0) = 0, \qquad u(r,\theta,l) = g(r), \qquad u(b,\theta,z) = 0. \tag{5.47}$$

The generalization to arbitrary $g(r,\theta)$ is left as Exercise 5. The case $g = h = 0$ is entirely similar to this one, and the case $f = g = 0$ will be discussed in §5.6. Of course the general case (5.46) can be solved by superposing the solutions to these three special cases.

Since our boundary conditions are independent of θ, we expect the solution to be independent of θ too (but see Exercise 5), so we apply separation of variables to $u(r,z) = R(r)Z(z)$ and find that

$$r^2R''(r) + rR'(r) + \mu^2r^2R(r) = 0, \qquad R(b) = 0.$$
$$Z''(z) - \mu^2 Z(z) = 0, \qquad Z(0) = 0.$$

The Sturm-Liouville problem for R has the eigenfunctions $J_0(\lambda_k r/b)$ where $\{\lambda_k\}_1^\infty$ are the positive zeros of J_0, with eigenvalues $\mu^2 = (\lambda_k/b)^2$. The corresponding solutions for Z are $\sinh(\lambda_k z/b)$. Hence, we obtain

$$u(r,z) = \sum_{k=1}^{\infty} a_k J_0\left(\frac{\lambda_k r}{b}\right) \sinh\frac{\lambda_k z}{b}.$$

To satisfy the boundary condition at $z = l$, we expand g in its Fourier-Bessel series,

$$g(r) = \sum_1^\infty c_k J_0\left(\frac{\lambda_k r}{b}\right),$$

and take

$$a_k = c_k \operatorname{csch}\frac{\lambda_k l}{b}.$$

Thus the Dirichlet problem with the special boundary conditions (5.47) is solved.

Example 3. If $g(r) \equiv 1$, the coefficient c_k was found at the end of §5.4 to be $2/\lambda_k J_1(\lambda_k)$. Therefore,

$$u(r,z) = 2\sum_1^\infty \frac{J_0(\lambda_k r/b)}{\lambda_k J_1(\lambda_k)} \frac{\sinh(\lambda_k z/b)}{\sinh(\lambda_k l/b)}.$$

EXERCISES

Exercises 1–4 deal with the heat equation in polar or cylindrical coordinates, in which we take the diffusivity coefficient k equal to 1.

1. A cylinder of radius b is initially at the constant temperature A. Find the temperatures in it at subsequent times if its ends are insulated and its circular surface obeys Newton's law of cooling, $u_r + cu = 0$ $(c > 0)$.

2. A circular cylinder of radius ρ initially is at the constant temperature A. At time $t = 0$ it is tightly wrapped in a sheath of the same material of thickness δ, thus forming a cylinder of radius $\rho + \delta$. The sheath is initially at temperature B, and its outside surface is maintained at temperature B. If the ends of the new, enlarged cylinder are insulated, find the temperature inside at subsequent times.
3. A cylindrical core of radius 1 is removed from a block of material whose temperature increases linearly from left to right. (Thus, if the cylinder occupies the region $x^2 + y^2 \leq 1$, the initial temperature is $ax + b$ for some constants a and b.) Find the subsequent temperatures in the core if
   a. it is completely insulated;
   b. its ends are insulated and its circular surface is maintained at temperature zero.
4. A cylindrical uranium rod of radius 1 generates heat within itself at a constant rate a. Its ends are insulated and its circular surface is immersed in a cooling bath at temperature zero. (Thus, $u_t = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + a$ and $u(1,t) = 0$.)
   a. Find the steady-state temperature $v(r)$ in the rod. (Hint: By symmetry, the steady-state temperature is independent of θ. Since $v_{rr} + r^{-1}v_r = r^{-1}(rv_r)_r$, the steady-state equation can be solved by integrating twice.)
   b. Find the temperature in the rod if its initial temperature is zero. (Hint: Again, u is independent of θ. Let $u = v + w$ with v as in part (a) and solve for w. Exercise 2, §5.4, is helpful.)
5. Solve problem (5.46) for a general $g(r,\theta)$ when $f = h = 0$. Prove that if g is independent of θ, your solution reduces to the one in the text.
6. Find the steady-state temperature in the cylinder $0 \leq r \leq 1$, $0 \leq z \leq 1$ when the circular surface is insulated, the bottom is kept at temperature 0, and the top is kept at temperature $f(r)$.
7. Analyze the vibrations of an elastic solid cylinder occupying the region $0 \leq r \leq 1$, $0 \leq z \leq 1$ in cylindrical coordinates if its top and bottom are held fixed, its circular surface is free, and the initial velocity u_t is zero. That is, find the general solution of

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz}),$$
$$u(r,\theta,0,t) = u(r,\theta,1,t) = u_r(1,\theta,z,t) = u_t(r,\theta,z,0) = 0.$$

8. Show that the eigenvalues of the Sturm-Liouville problem

$$[xf'(x)]' - \nu^2 x^{-1} f(x) + \lambda^2 x f(x) = 0 \quad (0 < a < x < b), \qquad f(a) = f(b) = 0$$

are the numbers λ^2 such that $J_\nu(\lambda a)Y_\nu(\lambda b) = J_\nu(\lambda b)Y_\nu(\lambda a)$. What are the corresponding eigenfunctions?

5.6 Variants of Bessel functions

Bessel functions arise in very diverse ways in physics and engineering, and it is often more convenient to work with certain functions related to J_ν and Y_ν rather than with J_ν and Y_ν themselves. In this section we discuss some of these related functions and the differential equations from which they arise.

Hankel functions

We saw in §5.3 that $J_\nu(x)$ behaves like a damped cosine when x is large and positive, and $Y_\nu(x)$ behaves like the corresponding damped sine:

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi), \qquad Y_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \sin(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi).$$

As it is often better to use e^{ix} and e^{-ix} instead of $\cos x$ and $\sin x$, we are led to consider the linear combinations

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x), \qquad H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x).$$

$H_\nu^{(1)}$ and $H_\nu^{(2)}$ are called the first and second **Hankel functions** or **Bessel functions of the third kind**. Their asymptotic behavior for large x is given by

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}} \left[\exp i(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\right]\left[1 + E_1(x)\right], \qquad |E_1(x)| \le \frac{C_\nu}{x},$$
$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}} \left[\exp i(-x + \tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi)\right]\left[1 + E_2(x)\right], \qquad |E_2(x)| \le \frac{C_\nu}{x}.$$

When stated in this form, with the error terms $E_1(x)$ and $E_2(x)$ multiplied by $e^{\pm ix}$, these formulas continue to hold for all *complex* $x = re^{i\theta}$ with $r \ge 1$ and $|\theta| < \pi$. (See Exercise 6, §8.6. There is a significant difference between $e^{ix} + E_1(x)$ and $e^{ix}[1 + E_1(x)]$ when x has an imaginary part, since then e^{ix} may be exponentially growing or decreasing.)

Modified Bessel functions

The **modified Bessel equation** is

$$x^2 f''(x) + xf'(x) - (x^2 + \nu^2)f(x) = 0. \tag{5.48}$$

It differs from the ordinary Bessel equation only in that x^2 is replaced by $-x^2$. In fact, it is the special case of the generalized Bessel equation

$$x^2 f''(x) + xf'(x) + (\mu^2 x^2 - \nu^2)f(x) = 0$$

in which $\mu = i$, so it can be reduced to Bessel's equation by the change of variable $x \to ix$. One solution of (5.48) is therefore $f(x) = J_\nu(ix)$, but it is more common use the constant multiple

$$I_\nu(x) = i^{-\nu} J_\nu(ix),$$

called the **modified Bessel function**. The reason is that, since $i^{\nu+2k} = i^\nu(-1)^k$,

$$I_\nu(x) = \sum_0^\infty \frac{1}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k},$$

which has the obvious advantage of being real when x and ν are real.

For a second independent solution of (5.48), one could use $I_{-\nu}(x)$ when ν is not an integer, or $Y_\nu(ix)$ for arbitrary ν. However, the standard choice is the function

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}.$$

Just as with Y_ν, this formula is well-defined whenever ν is not an integer and can be evaluated by l'Hopital's rule when ν is an integer, and it defines a solution of (5.48), independent of I_ν, for all ν. See Lebedev [36], §5.7, or Watson [55], §3.7.

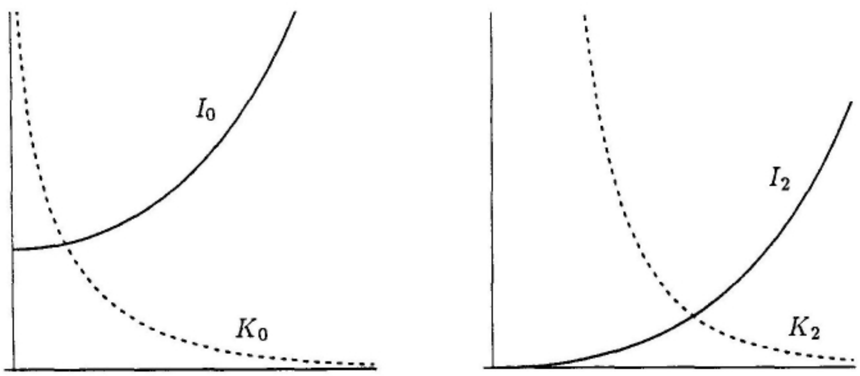


FIGURE 5.3. Graphs of some modified Bessel functions on the interval $0 \le x \le 3$. Left: I_0 (solid) and K_0 (dashed). Right: I_2 (solid) and K_2 (dashed).

The reason for choosing K_ν as the second independent solution of (5.48), like the reason for choosing Y_ν as the second solution of Bessel's equation, has to do with its asymptotic behavior for large x. Indeed, if we make the change of variable $f(x) = x^{-1/2}g(x)$ in (5.48), as we did with Bessel's equation in §5.3, we obtain

$$g''(x) - g(x) + \frac{\frac{1}{4} - \nu^2}{x^2} g(x) = 0.$$

When $x \gg 1$ we expect the last term in this equation to be negligibly small in comparison to the other two, so the solutions should look like the solutions of

$g'' - g = 0$, namely, $ae^x + be^{-x}$. Now, all the latter functions grow exponentially as $x \to +\infty$ except for the ones with $a = 0$, which decay exponentially; so we expect something similar to happen with solutions of (5.48). This is indeed the case, and K_ν is singled out as the only solution of (5.48) that tends to 0 rather than ∞ as $x \to +\infty$. More precisely, we have the following asymptotic formulas:

$$I_\nu(x) = \frac{1}{\sqrt{2\pi x}} e^x \left[1 + E_1(x)\right], \qquad K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + E_2(x)\right] \qquad (x \geq 1),$$

where, as usual, $E_1(x)$ and $E_2(x)$ are bounded by a constant, depending on ν, times x^{-1}. See Exercise 7, §8.6.

The asymptotic behavior of the modified Bessel functions as $x \to 0$ is just like that of the ordinary Bessel functions, except for constant factors. That is, if $\nu > 0$, $I_\nu(x) \sim c_\nu x^\nu$ and $K_\nu(x) \sim c'_\nu x^{-\nu}$, whereas $I_0(x) \sim 1$ and $K_0(x) \sim c \log x$. Thus, when $\nu \geq 0$, I_ν and its constant multiples are the only solutions of (5.48) that do not blow up as $x \to 0$. See Figure 5.3 on the previous page.

We met I_ν briefly in §5.4 when we were showing that certain Sturm-Liouville problems had no negative eigenvalues. Let us now display a situation where I_ν enters the solution in a positive way, namely, the Dirichlet problem in a right circular cylinder. We solved part of this problem in §5.5, and we now deal with the remaining case:

$$\begin{aligned} u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} &= 0 \quad \text{for } 0 \leq r < b,\ 0 < z < l, \\ u(r,\theta,0) = u(r,\theta,l) = 0, &\qquad u(b,\theta,z) = h(\theta,z). \end{aligned} \tag{5.49}$$

For simplicity we shall assume that h is independent of θ and leave the general case as Exercise 1. When the boundary conditions are independent of θ, the solution u will be too; so just as before, we try $u(r,z) = R(r)Z(z)$ and arrive at

$$r^2R''(r) + rR(r) + \mu^2r^2R(r) = 0,$$
$$Z''(z) - \mu^2 Z(z) = 0, \qquad Z(0) = Z(l) = 0.$$

The boundary conditions on Z now force $-\mu^2 = (n\pi/l)^2$ and $Z(z) = \sin(n\pi z/l)$, so the equation for R becomes

$$r^2R''(r) + rR'(r) - (n\pi r/l)^2 R(r) = 0,$$

which reduces under that change of variable $x = n\pi r/l$ to the modified Bessel equation of order zero. Of course $R(r)$ cannot blow up as $r \to 0$, so $R(r) = I_0(n\pi r/l)$. Hence, by superposition we arrive at

$$u(r,z) = \sum_1^\infty a_n I_0\left(\frac{n\pi r}{l}\right) \sin\frac{n\pi z}{l}.$$

To satisfy the boundary condition $u(b,z) = h(z)$, we have merely to expand h in its Fourier sine series on $[0,l]$ and match up coefficients:

$$a_n = \left[I_0\left(\frac{n\pi b}{l}\right)\right]^{-1} \frac{2}{l} \int_0^l f(z) \sin\frac{n\pi z}{l}\, dz.$$

Equations reducible to Bessel's equation

A large number of ordinary differential equations can be transformed into Bessel's equation by appropriate changes of independent and dependent variables. Here is one class of such equations.

Theorem 5.5. *Consider the equation*

$$x^p f''(x) + px^{p-1} f'(x) + (ax^q + bx^{p-2}) f(x) = 0, \tag{5.50}$$

where $(1-p)^2 - 4b \geq 0$ *and* $q - p + 2 > 0$. *Let*

$$\alpha = \frac{1-p}{2}, \qquad \beta = \frac{q-p+2}{2}, \qquad \lambda = \frac{2\sqrt{|a|}}{q-p+2}, \qquad \nu = \frac{\sqrt{(1-p)^2 - 4b}}{q-p+2}.$$

If $a > 0$, *the general solution of (5.50) is*

$$f(x) = x^\alpha \left[c_1 J_\nu(\lambda x^\beta) + c_2 Y_\nu(\lambda x^\beta) \right],$$

whereas if $a < 0$, *the general solution of (5.50) is*

$$f(x) = x^\alpha \left[c_1 I_\nu(\lambda x^\beta) + c_2 K_\nu(\lambda x^\beta) \right].$$

The proof, simple in principle but tedious to write out, consists merely of substituting $f(x) = x^\alpha g(x)$ and $x = (y/\lambda)^{1/\beta}$ into (5.50) and using the chain rule to reduce the resulting equation to the Bessel equation or modified Bessel equation relating g and y. Instead of presenting the messy details, let us look at some examples.

Example 1. If $p = q = a = 1$ and $b = -\nu^2$, (5.50) is just Bessel's equation of order ν (after multiplying through by x).

Example 2. If $p = q = b = 0$ and $a = 1$, (5.50) is the equation $f'' + f = 0$, whose solutions are linear combinations of $\cos x$ and $\sin x$. The solutions given by Theorem 5.5 are linear combinations of $x^{1/2} J_{1/2}(x)$ and $x^{1/2} Y_{1/2}(x)$; by (5.19) and the fact that $Y_{1/2} = -J_{-1/2}$, these are just what they should be.

Example 3. If $p = b = 0$ and $a = -1$, and $q = 1$, (5.50) becomes the **Airy equation** $f''(x) - xf(x) = 0$, which arises in the study of diffraction and related phenomena in optics. In Theorem 5.5 we have $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $\lambda = \frac{2}{3}$, and $\nu = \frac{1}{3}$, so the solutions are

$$f(x) = x^{1/2} \left[c_1 I_{1/3}(\tfrac{2}{3} x^{3/2}) + c_2 K_{1/3}(\tfrac{2}{3} x^{3/2}) \right]. \tag{5.51}$$

On the other hand, it is not hard to solve the Airy equation directly by assuming the solution to be of the form $f(x) = \sum a_n x^n$ and determining the coefficients a_n. We leave it as an exercise for the reader to do this and compare the results with the formula (5.51).

Example 4. As a final example, let us consider spherically symmetric waves in three dimensions, that is, solutions of the wave equation $u_{tt} = c^2\nabla^2 u$ of the form

$$u(x,y,z,t) = v(r,t), \quad \text{where } r = \sqrt{x^2+y^2+z^2}.$$

From the formula for the Laplacian in spherical coordinates (see Appendix 4), we have $\nabla^2 u = v_{rr} + 2r^{-1}v_r$, so the wave equation becomes

$$v_{tt} = c^2(v_{rr} + 2r^{-1}v_r). \tag{5.52}$$

Separation of variables, with $v(r,t) = R(r)T(t)$, leads to the equations

$$T''(t) + c^2\lambda^2 T(t) = 0, \qquad R''(r) + 2r^{-1}R'(r) + \lambda^2 R(r) = 0.$$

After being multiplied through by r^2, the equation for R is of the form (5.50) with $p = q = 2$, $a = \lambda^2$, and $b = 0$. Hence, by Theorem 5.5, the solutions are

$$R(r) = c_1 r^{-1/2} J_{1/2}(\lambda r) + c_2 r^{-1/2} Y_{1/2}(\lambda r) = \frac{a_1 \sin\lambda r + a_2 \cos\lambda r}{r}.$$

We therefore obtain as solutions of the spherical wave equation (5.52) the functions

$$v(r,t) = \frac{a_1 \sin\lambda r + a_2\cos\lambda r}{r}(b_1\cos\lambda ct + b_2 \sin\lambda ct) \tag{5.53}$$

and their superpositions obtained from taking different values of λ. Of course, if we want solutions that are nonsingular at the origin we must take $a_2 = 0$. This result can be generalized to arbitrary waves in spherical coordinates (without special symmetry properties), and the answer turns out to involve the Bessel functions J_ν where ν is an arbitrary half-integer. We shall work this out in Chapter 6.

We can now construct a model for vibrations of air in a conical pipe such as an oboe or saxophone. We place the vertex of the cone at the origin and take the length of the pipe, measured along the slanting side, to be l. We also assume that the vibrations depend only on the distance from the vertex, so that equation (5.52) applies. Here the simplest interpretation of v is as the so-called *condensation,* which is the change in the air pressure p relative to the ambient pressure p_0: $v = (p - p_0)/p_0$. (In analyzing the vibrations of a cylindrical pipe by the 1-dimensional wave equation in §4.2, we took the solution u to represent the displacement of the air. At least for small vibrations, the condensation in this situation is given by $v = -u_x$. If u satisfies the wave equation then so does u_x; so these descriptions are equivalent, although the boundary conditions look different. In higher dimensions the displacement is a vector quantity and so is more complicated to study. See Ingard [32] or Taylor [51].) The pressure must be equal to the ambient pressure at the open end of the pipe, so $v(l,t) = 0$, and it must be finite at the vertex. Therefore, $a_2 = 0$ and $\lambda = n\pi/l$ in (5.53), where n is a positive integer, and we obtain the general solution

$$v(r,t) = \sum_1^\infty \left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\frac{1}{r}\sin\frac{n\pi r}{l}. \tag{5.54}$$

Initial conditions $v(r,0) = f(r)$ and $v_t(r,0) = g(r)$ can be satisfied by expanding $rf(r)$ and $rg(r)$ in their Fourier sine series on $[0,l]$; see Exercise 3. Note that the allowable frequencies are the integer multiples of the fundamental frequency $l/2c$, just as in an open cylindrical pipe.

EXERCISES

1. Solve (5.49) for a general $h(\theta, z)$.
2. Find the general solution of the Airy equation $f''(x) - xf(x) = 0$ by assuming that $f(x) = \sum_0^\infty a_n x^n$ and determining the coefficients a_n. Since $x = 0$ is a regular point of the differential equation, a_0 and a_1 can be chosen arbitrarily; show that the solution f_1 with $a_0 = 1$, $a_1 = 0$ and the solution f_2 with $a_0 = 0$, $a_1 = 1$ are given by

$$f_1(x) = \Gamma(\tfrac{2}{3})3^{-1/3}x^{1/2}I_{-1/3}(\tfrac{2}{3}x^{3/2}), \qquad f_2(x) = \Gamma(\tfrac{4}{3})3^{1/3}x^{1/2}I_{1/3}(\tfrac{2}{3}x^{3/2}).$$

3. Determine the coefficients a_n and b_n in (5.54) if $v(r,0) = 0$ and $v_t(r,0) = l - r$.
4. A flexible cable hangs from a hook. (Assume the cable is on the z-axis, with bottom at $z = 0$ and top at $z = l$.) Since the tension at the point z on the cable is proportional to the weight of the portion of the cable below z, i.e., proportional to z, the appropriate wave equation to describe oscillations of the cable is $u_{tt} = c^2(zu_z)_z$ where u is the displacement. Since the top of the cable is fixed, $u(l,t) = 0$; and obviously the displacement $u(0,t)$ at the bottom must be finite. Find the general solution of this boundary value problem. (Hint: Use Theorem 5.5 and Exercise 9, §5.4.)

CHAPTER 6
ORTHOGONAL POLYNOMIALS

Some of the most useful orthogonal bases for L^2 spaces consist of polynomial functions. This chapter is a brief introduction to the most important of these orthogonal systems of polynomials; the last section also contains a discussion of some other interesting orthogonal bases.

6.1 Introduction

Let (a, b) be any open interval in $\mathbf{R}$, finite or infinite, and let $w(x)$ be a positive function on (a, b) such that the integrals $\int_a^b x^n w(x)\,dx$ $(n = 0, 1, 2, \ldots)$ are all absolutely convergent. Then there is a unique sequence $\{p_n\}_0^\infty$ of polynomials of the form

$$p_0(x) = 1, \qquad p_1(x) = x + a_0,$$
$$p_2(x) = x^2 + b_1 x + b_0, \qquad p_3(x) = x^3 + c_2 x^2 + c_1 x + c_0, \ldots$$

which is orthogonal with respect to the weight function w on (a, b). Indeed, the constant a_0 is fixed by the requirement that p_1 should be orthogonal to p_0:

$$0 = \langle p_1, p_0 \rangle_w = \int_a^b (x + a_0) w(x)\,dx \quad \Longrightarrow \quad a_0 = -\frac{\int_a^b x w(x)\,dx}{\int_a^b w(x)\,dx}.$$

Once a_0, and hence p_1, is known, the orthogonality conditions $\langle p_2, p_1 \rangle_w = 0$ and $\langle p_2, p_0 \rangle_w = 0$ give two linear equations that can be solved for the two constants b_0 and b_1 in p_2. Once these have been found, the three equations $\langle p_3, p_2 \rangle_w = \langle p_3, p_1 \rangle_w = \langle p_3, p_0 \rangle_w = 0$ can be solved for the constants c_0, c_1, c_2 in p_3. Continuing in this way, we see that the coefficients of all the polynomials p_n are determined by the orthogonality conditions. (It is not difficult to show by induction that the systems of linear equations determining the coefficients all have unique solutions, and to construct a recursive formula for the solutions.)

In short, associated to each weight function w on an interval (a, b) as above, there is a unique sequence $\{p_n\}_0^\infty$ of polynomials determined by the requirements that

(i) p_n is a polynomial of degree n,
(ii) $\langle p_n, p_m \rangle_w = 0$ for all $n \neq m$,
(iii) the coefficient of x^n in p_n is 1.
If we keep conditions (i) and (ii) but drop condition (iii), we have the freedom to multiply each p_n by an arbitrary nonzero constant c_n, and c_n can be chosen to make p_n satisfy another auxiliary condition in place of (iii). For example, it can be chosen so as to make $\|p_n\|_w = 1$ or to fix the value of p_n at some point.

Before proceeding, let us point out one simple fact that will be used repeatedly in this chapter.

Lemma 6.1. *Suppose $\{p_n\}_0^\infty$ is a sequence of polynomials such that p_n is of (exact) degree n for all n. Then every polynomial of degree k ($k = 0, 1, 2, \ldots$) is a linear combination of $p_0, \ldots, p_k$.*

Proof: If f is a polynomial of degree k, choose the constant c_k so that f and $c_k p_k$ have the same coefficient of x^k. Then $f - p_k c_k$ is a polynomial of degree $k-1$, so we can choose c_{k-1} so that $f - c_k p_k$ and $c_{k-1} p_{k-1}$ have the same coefficient of x^{k-1}. Then $f - c_k p_k - c_{k-1} p_{k-1}$ is a polynomial of degree $k-2$, and we can proceed inductively to choose $c_{k-2}, \ldots, c_0$ so that $f - \sum_0^k c_n p_n = 0$. ∎

The classical orthogonal polynomials we shall be studying in this chapter are eigenfunctions for certain singular Sturm-Liouville problems. We could proceed as we did in the case of Bessel functions, by first writing down the differential equation to be solved, finding its complete solution by the method of power series, and then singling out the polynomial solutions for special attention. However, since our aim here is to develop the basic properties of these polynomials as quickly and cleanly as possible, we have chosen to relegate these calculations to the exercises and to adopt a more direct approach. In each of the classical systems the polynomials p_n can be defined by a formula of the form

$$p_n(x) = \frac{C_n}{w(x)} \frac{d^n}{dx^n} \left[w(x) P(x)^n \right] \tag{6.1}$$

where C_n is a constant, $w(x)$ is the weight function with respect to which the p_n's are orthogonal, and $P(x)$ is a certain fixed polynomial. These formulas are known as **Rodrigues formulas** (the original formula of Rodrigues being the one pertaining to Legendre polynomials). From (6.1) it is easy to prove the orthogonality relations for the p_n's, to derive the differential equation that they satisfy, and to find their normalization constants. (The constants C_n in (6.1) are firmly fixed by tradition in each case, and they usually are not chosen to make $\|p_n\|_w = 1$. Hence, in order to expand general functions in terms of the basis $\{p_n\}$, it is necessary to know $\|p_n\|_w$.)

Since the Sturm-Liouville problems leading to the classical orthogonal polynomials are all singular, the general Sturm-Liouville theory does not guarantee that these orthogonal systems are complete. However, they *are* complete, and we shall establish this by invoking some theorems from Chapter 7. (The results in

Chapter 7 do not depend on the material in Chapter 6, so the reader is free to skip ahead and read them at any time.) We shall also derive generating functions for these polynomials and sketch some of their applications. Much more can be said, but for a more complete discussion we refer the reader to the books of Erdélyi et al. [21], Hochstadt [30], Lebedev [36], Rainville [44], and Szegö [50].

EXERCISE

1. Let $\{p_n\}_0^\infty$ be an orthogonal set in $L^2_w(a,b)$, where p_n is a polynomial of degree n.
   a. Fix a value of n. Let $x_1, x_2, \ldots x_k$ be the points in (a,b) where p_n changes sign, i.e., where its graph crosses the x-axis, and let $q(x) = \prod_1^k (x - x_j)$. Show that $p_n q$ never changes sign on (a,b) and hence that $\langle p_n, q\rangle_w \neq 0$.
   b. Show that the number k of sign changes in part (a) is at least n. (Hint: If $k < n$ then $\langle p_n, q\rangle_w = 0$. Why?)
   c. Conclude that p_n has exactly n distinct zeros, all of which lie in (a,b). (Geometrically, this indicates that p_n becomes more and more oscillatory on (a,b) as $n \to \infty$, rather like $\sin nx$.)

6.2 Legendre polynomials

The nth **Legendre polynomial**, denoted by P_n, is defined by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{6.2}$$

The function $(x^2-1)^n$ is a polynomial of degree $2n$ with leading term x^{2n}, so P_n is a polynomial of degree n. For the first few values of n we have

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \tfrac{1}{2}(3x^2-1),$$
$$P_3(x) = \tfrac{1}{2}(5x^3-3x), \qquad P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3).$$

See Figure 6.1. The coefficients of P_n can be calculated by using the binomial theorem (Exercise 1), but all we shall need is the leading one:

$$\begin{aligned} P_n(x) &= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^{2n} + \cdots) = \frac{1}{2^n n!}\Big[(2n)(2n-1)\cdots(n+1)x^n + \cdots\Big] \\ &= \frac{(2n)!}{2^n (n!)^2}x^n + \cdots, \end{aligned} \tag{6.3}$$

where the dots denote terms of lower degree.

We begin by establishing the orthogonality properties of the Legendre polynomials. In what follows we shall be working in the space $L^2(-1,1)$ (with weight function $w(x) \equiv 1$), and $\langle\, ,\, \rangle$ will denote the inner product in this space.

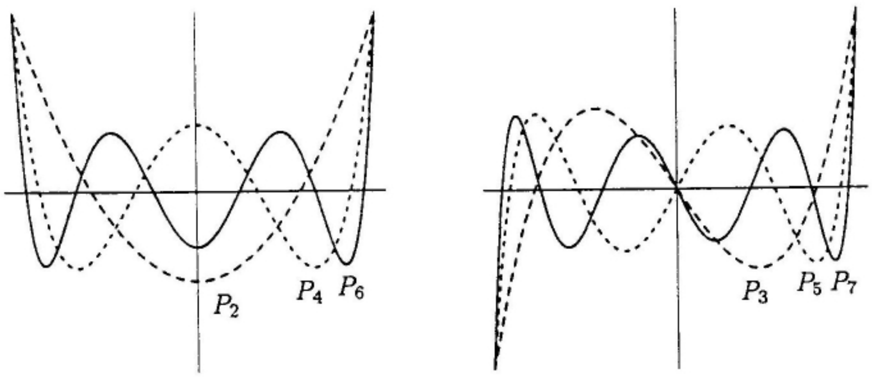


FIGURE 6.1. Graphs of some Legendre polynomials on the interval $-1 \leq x \leq 1$. Left: P_2 (long dashes), P_4 (short dashes), and P_6 (solid). Right: P_3 (long dashes), P_5 (short dashes), and P_7 (solid).

Theorem 6.1. *The Legendre polynomials $\{P_n\}_0^\infty$ are orthogonal in $L^2(-1,1)$, and*

$$\|P_n\|^2 = \frac{2}{2n+1}. \tag{6.4}$$

Proof: The key observation is that if f is any function of class $C^{(n)}$ on $[-1,1]$, we have

$$2^n n! \langle f, P_n \rangle = \int_{-1}^{1} f(x) \frac{d^n}{dx^n}(x^2-1)^n \, dx = (-1)^n \int_{-1}^{1} f^{(n)}(x)(x^2-1)^n \, dx. \tag{6.5}$$

The second of these equations follows by an n-fold integration by parts; the endpoint terms are all zero because the function $(x^2-1)^n = (x-1)^n(x+1)^n$ vanishes to nth order at $x = \pm 1$ and hence its first $n-1$ derivatives all vanish at $x = \pm 1$. If f is a polynomial of degree less than n then $f^{(n)} \equiv 0$, so $\langle f, P_n \rangle = 0$. In particular, this is true of $P_0, \ldots, P_{n-1}$, so $\langle P_m, P_n \rangle = 0$ for $m < n$. By the same reasoning with m and n interchanged, we also have $\langle P_m, P_n \rangle = 0$ for $m > n$, so the P_n's are mutually orthogonal.

On the other hand, if we take $f = P_n$, by (6.3) we have

$$f^{(n)}(x) \equiv \frac{(2n)!}{2^n n!} = \frac{1 \cdot 2 \cdot 3 \cdots (2n)}{2 \cdot 4 \cdots (2n)} = 1 \cdot 3 \cdot 5 \cdots (2n-1),$$

so by (6.5),

$$\|P_n\|^2 = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} \int_{-1}^{1} (1-x^2)^n \, dx.$$

But by the substitution $x = \sqrt{y}$ and the formula for the beta integral (Appendix 3),

$$\int_{-1}^{1}(1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx = \int_0^1 (1-y)^n y^{-1/2} dy = \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}$$
$$= \frac{n!\sqrt{\pi}}{\Gamma(n+\frac{3}{2})} = \frac{n!}{(\frac{1}{2})(\frac{3}{2})\cdots(n+\frac{1}{2})} = \frac{2^{n+1}n!}{1\cdot 3\cdot 5\cdots(2n+1)},$$

which proves (6.4). ∎

We next derive the differential equation satisfied by the Legendre polynomials.

Theorem 6.2. *For all $n \geq 0$ we have*

$$\left[(1-x^2)P_n'(x)\right]' + n(n+1)P_n(x) = 0. \tag{6.6}$$

Proof: Let $g(x) = \left[(1-x^2)P_n'(x)\right]'$. Since P_n' is a polynomial of degree $n-1$, x^2P_n' is of degree $n+1$, and hence g is of degree n. In fact, by (6.3), its leading term is

$$\frac{(2n)!}{2^n(n!)^2}\frac{d}{dx}\left[(-x^2)(nx^{n-1})\right] = -n(n+1)\frac{(2n)!}{2^n(n!)^2}x^n.$$

Thus, in view of (6.3), $g + n(n+1)P_n$ is a polynomial of degree $n-1$, so by Lemma 6.1 it is a linear combination of $P_0, \dots, P_{n-1}$:

$$g(x) + n(n+1)P_n(x) = \left[(1-x^2)P_n'(x)\right]' + n(n+1)P_n(x) = \sum_0^{n-1} c_jP_j(x).$$

By orthogonality, the coefficients c_j can be calculated in terms of inner products:

$$c_j = \frac{\langle g + n(n+1)P_n, P_j\rangle}{\|P_j\|^2} = \frac{\langle g, P_j\rangle + n(n+1)\langle P_n, P_j\rangle}{\|P_j\|^2}.$$

Now $\langle P_n, P_j\rangle = 0$ for $j < n$, and

$$\langle g, P_j\rangle = \int_{-1}^{1}\left[(1-x^2)P_n'(x)\right]' P_j(x)\,dx.$$

After two integrations by parts, in which the boundary terms vanish since $x^2 - 1 = 0$ when $x = \pm 1$,

$$\langle g, P_j\rangle = \int_{-1}^{1} P_n(x)\left[(1-x^2)P_j'(x)\right]' dx.$$

But $\left[(1-x^2)P_j'(x)\right]'$ is a polynomial of degree j, hence is a linear combination of $P_0, \dots, P_j$, hence is orthogonal to P_n. Therefore, $c_j = 0$ for all $j < n$, and we are done. ∎

Theorem 6.2 says that the Legendre polynomials are eigenfunctions for the **Legendre equation**

$$\left[(1-x^2)y'\right]' + \lambda y = 0, \tag{6.7}$$

the eigenvalue for P_n being $n(n+1)$. This is an equation of Sturm-Liouville type on the interval $(-1,1)$, but it is singular since the leading coefficient $1-x^2$ vanishes at both endpoints. To arrive at the appropriate boundary conditions to define a Sturm-Liouville problem, one must examine the behavior of the solutions of (6.7) near $x = \pm 1$.

Briefly, the situation is as follows. The points $x = \pm 1$ are regular singular points for equation (6.7), and it is easily verified that the characteristic exponents at each of these singular points are both zero. Hence, for any λ, equation (6.7) will have one nontrivial solution that is analytic at $x = 1$, whereas any second independent solution will have a logarithmic singularity there; the same is true at $x = -1$. We may therefore impose boundary conditions on (6.7) by requiring that the solutions have no singularity at $x = \pm 1$, a requirement that can be phrased as follows:

$$\lim_{x \to 1} y(x) \text{ and } \lim_{x \to -1} y(x) \text{ exist.} \tag{6.8}$$

The Legendre polynomials are then eigenfunctions for the Sturm-Liouville problem defined by (6.7) and (6.8). We shall now establish the completeness of the Legendre polynomials, which implies in particular that there are no other eigenfunctions for (6.7) and (6.8).

Theorem 6.3. *$\{P_n\}_0^\infty$ is an orthogonal basis for $L^2(-1,1)$.*

Proof: Suppose $f \in L^2(-1,1)$ is orthogonal to all the P_n's, and hence (by Lemma 6.1) orthogonal to every polynomial. Given a small positive number ϵ, there is a continuous function g on $[-1,1]$ such that $\|f-g\| < \frac{1}{2}\epsilon$ (Theorem 3.3, §3.3). By the Weierstrass approximation theorem, which we shall prove in §7.1, there is a polynomial P such that $|P(x) - g(x)| < \frac{1}{4}\epsilon$ for all $x \in [-1,1]$, and hence such that

$$\|P-g\| = \left(\int_{-1}^{1} |P(x)-g(x)|^2 dx\right)^{1/2} < \tfrac{1}{4}\epsilon\sqrt{2} < \tfrac{1}{2}\epsilon.$$

But then

$$\|f\|^2 = \langle f, f\rangle = \langle f-g, f\rangle + \langle g-P, f\rangle + \langle P, f\rangle,$$

and since $\langle P, f\rangle = 0$ by hypothesis, the Cauchy-Schwarz inequality yields

$$\|f\|^2 \le \|f-g\|\,\|f\| + \|g-P\|\,\|f\| < \epsilon\|f\|,$$

so that $\|f\| < \epsilon$. Since ϵ is arbitrary, $f = 0$. ∎

In view of Theorem 6.1, the expansion of a function $f \in L^2(-1,1)$ in terms of Legendre polynomials is given by

$$f = \sum_0^\infty c_n P_n, \quad \text{where } c_n = \frac{2n+1}{2}\langle f, P_n \rangle = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx. \tag{6.9}$$

The series $\sum c_n P_n$ converges in norm; it can also be shown to converge pointwise provided that f is piecewise smooth, just as in the case of Fourier series.

A related type of expansion is sometimes useful. We observe that $(x^2-1)^n$ is an even function of x, so that its nth derivative $2^n n! P_n(x)$ is even or odd according as n is even or odd. Therefore, just as we passed from Fourier series on $[-\pi,\pi]$ to Fourier cosine and sine series on $[0,\pi]$, we can pass from series of Legendre polynomials on $[-1,1]$ to series of even or odd Legendre polynomials on $[0,1]$. The result is the following.

Theorem 6.4. *$\{P_{2n}\}_{n=0}^\infty$ and $\{P_{2n+1}\}_{n=0}^\infty$ are orthogonal bases for $L^2(0,1)$. The norm of P_k in $L^2(0,1)$ is $(2k+1)^{-1/2}$.*

The details of the proof are left to the reader (Exercise 11). The functions P_{2n} and P_{2n+1} are the eigenfunctions of the Sturm-Liouville problems on $(0,1)$ defined by the Legendre equation (6.7) and the boundary conditions

$$\lim_{x\to 1} y(x) \text{ exists}, \quad y'(0)=0 \quad (\text{for } P_{2n}),$$
$$\lim_{x\to 1} y(x) \text{ exists}, \quad y(0)=0 \quad (\text{for } P_{2n+1}).$$

The following identity gives the generating function for the Legendre polynomials. We shall derive it by means of contour integrals. Another approach (see Rainville [44] and Walker [53]) is to take this identity as a *definition* of the Legendre polynomials and develop the theory from there; in fact, this is what Legendre did originally.

Theorem 6.5. *For $-1 \le x \le 1$ and $|z| < 1$ we have*

$$\sum_0^\infty P_n(x)z^n = (1-2xz+z^2)^{-1/2}. \tag{6.10}$$

(Here z may be complex, and the principal branch of the square root function is used on the right.)

Proof: Given $x \in [-1,1]$, let γ denote the circle of radius 1 about x in the complex plane. Applying the Cauchy formula for derivatives (Appendix 2) to the formula (6.2), we have

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2\pi i}\int_\gamma \frac{(\zeta^2-1)^n}{2^n(\zeta-x)^{n+1}}\,d\zeta.$$

Thus, if $|z|$ is small enough so that the geometric series $\sum\left[z(\zeta^2-1)/2(\zeta-x)\right]^n$ converges uniformly for $\zeta \in \gamma$ ($|z| < \frac{2}{5}$ is good enough),

$$\sum_0^\infty P_n(x)z^n = \frac{1}{2\pi i}\int_\gamma \sum_0^\infty \left(\frac{z}{2}\right)^n \frac{(\zeta^2-1)^n}{(\zeta-x)^{n+1}}\,d\zeta$$
$$= \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta-x}\left[1-\frac{z(\zeta^2-1)}{2(\zeta-x)}\right]^{-1} d\zeta$$
$$= \frac{1}{2\pi i}\int_\gamma \frac{2\,d\zeta}{z-2x+2\zeta-z\zeta^2}.$$

The zeros of $z-2x+2\zeta-z\zeta^2$, as a function of ζ, occur at

$$\zeta_1 = \frac{1-\sqrt{1-2xz+z^2}}{z} \quad \text{and} \quad \zeta_2 = \frac{1+\sqrt{1-2xz+z^2}}{z}.$$

When $|z|$ is small, $\sqrt{1-2xz+z^2}$ is approximately $1-xz$ (by the tangent line approximation), so ζ_1 is close to x while ζ_2 is very large. In particular, ζ_1 is inside the circle γ and ζ_2 is outside, so a simple calculation with the residue theorem gives

$$\sum_0^\infty P_n(x)z^n = \operatorname{Res}_{\zeta=\zeta_1} \frac{2}{z-2x+2\zeta-z\zeta^2} = (1-2xz+z^2)^{-1/2}.$$

Thus (6.10) is proved assuming that $|z|$ is sufficiently small. But then the series on the left of (6.10) is the Taylor series of the analytic function on the right, and its radius of convergence is the distance from the origin to the singularities of the latter function at $z = x \pm i\sqrt{1-x^2}$, namely, 1. The formula is therefore valid for all z such that $|z| < 1$. ∎

Corollary 6.1. *For all n we have $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.*

Proof: On setting $x = \pm 1$ in (6.10) we have

$$\sum_0^\infty P_n(1)z^n = \frac{1}{1-z}, \qquad \sum_0^\infty P_n(-1)z^n = \frac{1}{1+z}.$$

But the Taylor series of the functions $(1-z)^{-1}$ and $(1+z)^{-1}$ are just the geometric series $\sum z^n$ and $\sum(-1)^n z^n$. The result follows by comparing coefficients of z^n. ∎

Formula (6.10) has an interesting physical interpretation. If a charge (or mass) is located at the point $\mathbf{a}$ in $\mathbf{R}^3$, the induced electrostatic (or gravitational) potential at the point $\mathbf{x}$ is, up to a constant multiple, $|\mathbf{x}-\mathbf{a}|^{-1}$. Suppose that $\mathbf{a}$ is at a unit distance from the origin; let $r = |\mathbf{x}|$, and let θ be the angle between the vectors $\mathbf{x}$ and $\mathbf{a}$. Then by the geometric interpretation of the dot product,

$$|\mathbf{x}-\mathbf{a}|^{-1} = \left[(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})\right]^{-1/2} = (r^2 - 2r\cos\theta + 1)^{-1/2}.$$

Therefore, by (6.10), if $r < 1$ we have

$$|\mathbf{x} - \mathbf{a}|^{-1} = \sum_0^\infty P_n(\cos\theta) r^n.$$

That is, the Legendre polynomials give the expansion of the potential about the origin in powers of $r = |\mathbf{x}|$. Other applications of Legendre polynomials will be given in §6.2.

We conclude this section with a formula relating the Legendre polynomials and their derivatives.

Theorem 6.6. *For all $n \geq 1$ we have*

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).$$

Proof: The second derivative of $(x^2-1)^{n+1}$ is

$$\frac{d}{dx}\Big[2(n+1)x(x^2-1)^n\Big] = 2(n+1)\Big[(x^2-1)^n + 2nx^2(x^2-1)^{n-1}\Big],$$

and by writing $x^2 = (x^2-1)+1$ on the right we see that

$$\frac{d^2}{dx^2}(x^2-1)^{n+1} = 2(n+1)(2n+1)(x^2-1)^n + 4(n+1)n(x^2-1)^{n-1}.$$

Therefore, by formula (6.2),

$$\begin{aligned} P_{n+1}(x) &= \frac{1}{2^{n+1}(n+1)!}\frac{d^{n-1}}{dx^{n-1}}\frac{d^2}{dx^2}(x^2-1)^{n+1} \\ &= \frac{(2n+1)}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n + \frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1} \\ &= \frac{(2n+1)}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n + P_{n-1}(x). \end{aligned}$$

Differentiating both sides and applying formula (6.2) once more, we are done. ∎

Theorem 6.6 can also be stated as

$$\int P_n(x)\,dx = \frac{1}{2n+1}\Big[P_{n+1}(x) - P_{n-1}(x)\Big] + C \qquad (n \geq 1), \tag{6.11}$$

a useful integration formula. (This formula holds also for $n = 0$ if we set $P_{-1}(x) = 0$.)

EXERCISES

1. Show that
$$P_n(x) = \frac{1}{2^n} \sum_{j \le n/2} \frac{(-1)^j (2n-2j)!}{j!(n-j)!(n-2j)!} x^{n-2j}.$$
2. Deduce from Exercise 1 that
$$P_{2k-1}(0) = 0, \qquad P_{2k}(0) = \frac{(-1)^k (2k)!}{2^{2k}(k!)^2}.$$
3. Find the general solution of the Legendre equation
$$\left[(1-x^2)y'\right]' + \lambda y = 0 \qquad (-1 < x < 1)$$
where λ is an arbitrary complex number. To do this, rewrite the equation in the form
$$(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0$$
where ν is again an arbitrary complex number, set $y = \sum_0^\infty a_n x^n$, and determine the coefficients a_n recursively in terms of a_0 and a_1. Use the ratio test to verify that the resulting series converge on $(-1, 1)$.
4. With reference to Exercise 3, show that the Legendre equation has a polynomial solution precisely when ν is an integer, and that this solution is a constant multiple of P_ν if $\nu \ge 0$ or $P_{-\nu-1}$ if $\nu < 0$.
5. Show that the generating function $F(x,z) = (1 - 2xz + z^2)^{-1/2}$ of Theorem 6.5 satisfies $(1 - 2xz + z^2)(\partial F/\partial z) = (x - z)F$, and deduce the recursion formula
$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$
6. Expand x^2, x^3, and x^4 in series of Legendre polynomials. (Hint: No calculus is needed. Cf. the proof of Lemma 6.1.)
7. Let $f(x) = 1$ for $0 < x < 1$ and $f(x) = -1$ for $-1 < x < 0$. Expand f is a series of Legendre polynomials. (Hint: Use equation (6.11) and Exercise 2.)
8. Let $f(x) = x$ for $0 < x < 1$ and $f(x) = 0$ for $-1 < x < 0$. Expand f in a series of Legendre polynomials. (Hint: Use equation (6.11); you can leave the answer in terms of the numbers $P_n(0)$ or evaluate the latter by Exercise 2.)
9. As in the proof of Theorem 6.5, write
$$P_n(x) = \frac{1}{2\pi i} \int_\gamma \frac{(\zeta^2 - 1)^n}{2^n (\zeta - x)^{n+1}} \, d\zeta.$$
For $-1 < x < 1$, take γ to be the circle of radius $\sqrt{1-x^2}$ about x, and deduce *Laplace's integral formula*
$$P_n(x) = \frac{1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2} \cos\theta\right)^n d\theta.$$
10. Deduce from Exercise 9 that $|P_n(x)| \le 1$ for $-1 \le x \le 1$.
11. Deduce Theorem 6.4 from Theorem 6.3.

6.3 Spherical coordinates and Legendre functions

In this section we shall solve some boundary value problems involving the Laplacian in spherical coordinates. We recall (see Appendix 4) that the spherical coordinates of a point $\mathbf{x} = (x, y, z) \in \mathbf{R}^3$ are given by

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi,$$

and that the Laplacian in spherical coordinates is given by

$$\nabla^2 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin\phi}(u_\phi\sin\phi)_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}. \tag{6.12}$$

To begin with, we consider the Dirichlet problem for the unit ball in $\mathbf{R}^3$:

$$\nabla^2 u(r,\theta,\phi) = 0 \quad \text{for } r < 1, \qquad u(1,\theta,\phi) = f(\theta,\phi). \tag{6.13}$$

Applying the method of separation of variables, we look for solutions of $\nabla^2 u = 0$ in the form $u = R(r)\Theta(\theta)\Phi(\phi)$. Substituting this expression into the equation $\nabla^2 u = 0$ and rearranging the terms, we obtain

$$r^2\sin^2\phi\left[\frac{R''}{R} + \frac{2R'}{rR}\right] + \sin\phi\frac{(\Phi'\sin\phi)'}{\Phi} = -\frac{\Theta''}{\Theta}. \tag{6.14}$$

Both sides must equal a constant m^2. Thus $\Theta'' + m^2\Theta = 0$ and hence

$$\Theta(\theta) = ae^{im\theta} + be^{-im\theta}.$$

Since θ represents the longitude in spherical coordinates, Θ must be 2π-periodic; hence m must be an integer, which we may take to be nonnegative.

We now set the left side of (6.14) equal to m^2 and separate r and ϕ:

$$\frac{r^2R'' + 2rR'}{R} = \frac{m^2}{\sin^2\phi} - \frac{(\Phi'\sin\phi)'}{\Phi\sin\phi}.$$

Here both sides must equal a constant λ, and the equations for Φ and R can be written

$$\frac{(\Phi'\sin\phi)'}{\sin\phi} - \frac{m^2\Phi}{\sin^2\phi} + \lambda\Phi = 0, \tag{6.15}$$

$$r^2R'' + 2rR' - \lambda R = 0. \tag{6.16}$$

Equation (6.15) can be transformed into a close relative of the Legendre equation (6.7) by the substitution $s = \cos\phi$. (Recall that ϕ, the co-latitude, ranges over the interval $[0,\pi]$. The transformation $\phi \to s = \cos\phi$ is a one-to-one correspondence between $[0,\pi]$ and $[-1,1]$.) Indeed, if q is a quantity depending on s and hence on ϕ, we have

$$\frac{dq}{d\phi}=\frac{dq}{ds}\frac{ds}{d\phi}=-\sin\phi\frac{dq}{ds},\quad \text{or}\quad \frac{1}{\sin\phi}\frac{dq}{d\phi}=-\frac{dq}{ds}.$$

Hence, if we set

$$s=\cos\phi,\qquad S(s)=S(\cos\phi)=\Phi(\phi)$$

and note that $\sin^2\phi=1-s^2$, we have

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right)=\frac{d}{ds}\left((1-s^2)\frac{dS}{ds}\right).$$

Therefore, $\Phi(\phi)$ satisfies (6.15) if and only if $S(s)=\Phi(\arccos s)$ satisfies

$$\left[(1-s^2)S'\right]'-\frac{m^2S}{1-s^2}+\lambda S=0. \tag{6.17}$$

When $m=0$ this is just the Legendre equation (6.7). In general, (6.17) is called the **associated Legendre equation** of order m.

When m is a positive integer, as it is in our case, it is easy to find solutions of (6.17) in terms of the solutions of the ordinary Legendre equation

$$\left[(1-s^2)w'\right]'+\lambda w=0. \tag{6.18}$$

Indeed, let w be a solution of (6.18). If we apply the product rule for $(m+1)$th order derivatives,

$$(fg)^{(m+1)}=\sum_0^{m+1}\frac{(m+1)!}{k!(m+1-k)!}f^{(k)}g^{(m+1-k)}$$

to $f(s)=1-s^2$ and $g(s)=w'(s)$, we obtain

$$\left[(1-s^2)w'\right]^{(m+1)}=(1-s^2)w^{(m+2)}-2(m+1)sw^{(m+1)}-m(m+1)w^{(m)},$$

so by differentiating (6.18) m times we obtain

$$(1-s^2)w^{(m+2)}-2(m+1)sw^{(m+1)}-m(m+1)w^{(m)}+\lambda w^{(m)}=0. \tag{6.19}$$

Now let

$$S=(1-s^2)^{m/2}w^{(m)}.$$

We have

$$(1-s^2)S'=-ms(1-s^2)^{m/2}w^{(m)}+(1-s^2)^{(m/2)+1}w^{(m+1)},$$

and hence, after a straightforward calculation,

$$\Big[(1-s^2)S'\Big]' = (1-s^2)^{m/2} \times \left[(1-s^2)w^{(m+2)} - 2(m+1)sw^{(m+1)} + \frac{m^2w^{(m)}}{1-s^2} - m(m+1)w^{(m)}\right].$$

But in view of (6.19), this means that

$$\Big[(1-s^2)S'\Big]' = \frac{m^2S}{1-s^2} - \lambda S.$$

In other words, if w satisfies (6.18), then $S = (1-s^2)^{m/2}w^{(m)}$ satisfies (6.17).

In particular, if we take $\lambda = n(n+1)$ and take w to be the Legendre polynomial P_n, we obtain the **associated Legendre function** P_n^m:

$$P_n^m(s) = (1-s^2)^{m/2}\frac{d^mP_n(s)}{ds^m} = \frac{(1-s^2)^{m/2}}{2^nn!}\frac{d^{n+m}}{ds^{n+m}}(s^2-1)^n. \tag{6.20}$$

(*Note:* Some authors insert an extra factor of $(-1)^m$ into the definition of $P_n^m(s)$.) We observe that $P_n^m(s) \equiv 0$ when $m > n$, since P_n is a polynomial of degree n, so P_n^m is of interest only for $n \geq m$. But for $m = 1, 2, 3, \ldots$ and $n \geq m$, P_n^m is a solution of the boundary value problem

$$\begin{gathered}\Big[(1-s^2)y'\Big]' + \frac{m^2y}{1-x^2} + n(n+1)y = 0,\\ y(-1) = y(1) = 0.\end{gathered} \tag{6.21}$$

Theorem 6.7. *For each positive integer m, $\{P_n^m\}_{n=m}^\infty$ is an orthogonal basis for $L^2(-1,1)$, and*

$$\|P_n^m\|^2 = \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1}.$$

Proof: The orthogonality of P_n^m and $P_{n'}^m$ for $n \neq n'$ follows by the usual integration by parts from the fact that P_n^m satisfies (6.21); cf. the proof of Theorem 3.9 in §3.5. Also, from (6.20) we see that

$$P_{m+k}^m(s) = (1-s^2)^{m/2}q_k(s)$$

where q_k is a polynomial of degree k (we have set $n = m+k$ since $n \geq m$), and

$$\langle P_{m+k}^m, P_{m+k'}^m\rangle = \int_{-1}^1 q_k(s)q_{k'}(s)(1-s^2)^m\,ds.$$

That is, the polynomials q_k are orthogonal with respect to the weight function $w(s) = (1-s^2)^m$ on $(-1,1)$. The completeness of the set $\{q_k\}_0^\infty$ in $L^2_w(-1,1)$

follows from the Weierstrass approximation theorem, just as in the proof of Theorem 6.3. But then if $f \in L^2(-1,1)$ is orthogonal to all the P^m_{m+k} in $L^2(-1,1)$, the function $g(s) = (1-s^2)^{-m/2} f(s)$ will be orthogonal to all the q_k in $L^2_w(-1,1)$:

$$0 = \int_{-1}^{1} f(s) P^m_{m+k}(s)\,ds = \int_{-1}^{1} g(s) q_k(s)(1-s^2)^m\,ds.$$

It follows that $g = 0$ and hence $f = 0$, so the set $\{P^m_{m+k}\}_{k=0}^{\infty}$ is complete.

Finally, we compute the norm of P^m_n. To simplify the notation we fix n and set

$$y_m = P^m_n(s) \quad (m = 1, 2, \ldots, n), \qquad y_0 = P_n(s).$$

First, from (6.20) we have

$$\begin{aligned} y'_{m-1} &= \frac{d}{ds}\left[(1-s^2)^{(m-1)/2} \frac{d^{m-1}P_n(s)}{ds^{m-1}}\right] \\ &= -(m-1)s(1-s^2)^{(m-3)/2} \frac{d^{m-1}P_n(s)}{ds^{m-1}} + (1-s^2)^{(m-1)/2} \frac{d^m P_n(s)}{ds^m} \\ &= -(m-1)s(1-s^2)^{-1} y_{m-1} + (1-s^2)^{-1/2} y_m. \end{aligned}$$

In other words,

$$y_m = \sqrt{1-s^2}\, y'_{m-1} + \frac{(m-1)s}{\sqrt{1-s^2}} y_{m-1}.$$

Square both sides and integrate from -1 to 1:

$$\|y_m\|^2 = \int_{-1}^{1} \left[(1-s^2)(y'_{m-1})^2 + 2(m-1)s y_{m-1} y'_{m-1} + \frac{(m-1)^2 s^2}{1-s^2}(y_{m-1})^2\right] ds. \tag{6.22}$$

Now integrate the first two terms on the right by parts. For the first one, by (6.21) (with m replaced by $m-1$) we obtain

$$\begin{aligned} \int_{-1}^{1} (1-s^2)(y'_{m-1})^2\,ds &= -\int_{-1}^{1} y_{m-1}\left[(1-s^2) y'_{m-1}\right]'\,ds \\ &= \int_{-1}^{1} \left[n(n+1) - \frac{(m-1)^2}{1-s^2}\right](y_{m-1})^2\,ds, \end{aligned}$$

whereas for the second one, we have

$$\begin{aligned} \int_{-1}^{1} s y_{m-1} y'_{m-1}\,ds &= -\int_{-1}^{1} y_{m-1}[s y_{m-1}]'\,ds \\ &= -\int_{-1}^{1} s y_{m-1} y'_{m-1}\,ds - \int_{-1}^{1} (y_{m-1})^2\,ds, \end{aligned}$$

which implies that

$$2\int_{-1}^{1} s y_{m-1} y'_{m-1}\, ds = -\int_{-1}^{1} (y_{m-1})^2\, ds.$$

Substituting these results into (6.22), we find that

$$\|y_m\|^2 = [n(n+1) - m(m-1)] \int_{-1}^{1} (y_{m-1})^2\, ds = (n+m)(n-m+1)\|y_{m-1}\|^2.$$

It therefore follows by induction that

$$\|y_m\|^2 = (n+m)\cdots(n+2)(n+1)(n-m+1)\cdots(n-1)n\|y_0\|^2,$$

or, in other words,

$$\|P_n^m\|^2 = \frac{(n+m)!}{(n-m)!}\|P_n\|^2 = \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1},$$

where we have used Theorem 6.1 for the last equation. ∎

We now return to the Dirichlet problem (6.13). What we have found so far is that in the separated solution $u = R(r)\Theta(\theta)\Phi(\phi)$ of Laplace's equation, $\Theta(\theta)$ has the form $ae^{im\theta} + be^{-im\theta}$ with m a nonnegative integer, and $\Phi(\phi)$ has the form $y(\cos\phi)$ where y is a solution of the associated Legendre equation (6.17). Moreover, since we wish u to be a continuous function on the unit ball, $y(\pm 1)$ must be finite, and when $m > 0$, $y(\pm 1)$ must actually be zero. The reason is that the longitude θ is not well-defined along the z-axis, where $\cos\phi = \pm 1$, so the function $y(\cos\phi)(ae^{im\theta} + be^{-im\theta})$ will be discontinuous there unless $y(\pm 1) = 0$. The Legendre polynomials P_n (for $m = 0$) and the associated Legendre functions P_n^m (for $m > 0$) are solutions of these boundary value problems, and since they form complete orthogonal sets in $L^2(-1,1)$, there are no other independent solutions.

In particular, the eigenvalue λ in the Legendre equation must be of the form $n(n+1)$ where n is a nonnegative integer, so the equation (6.16) for R becomes

$$r^2R'' + 2rR' - n(n+1)R = 0. \tag{6.23}$$

This is an Euler equation, and its general solution is

$$R(r) = ar^n + br^{-n-1}.$$

Since we want the solution to be continuous at the origin, we must take $b = 0$.

In short, we have found the following family of solutions of Laplace's equation:

$$u_{mn}(r,\theta,\phi) = r^n e^{im\theta} P_n^{|m|}(\cos\phi) \qquad (n = 0,1,2,\ldots;\ |m| \le n).$$

Here it is understood that $P_n^0 = P_n$, and we are using the letter m in a slightly different way than we did before in order to list $e^{im\theta}$ and $e^{-im\theta}$ separately. We therefore hope to solve our original problem (6.13) by taking a superposition of these solutions:

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{mn} r^n e^{im\theta} P_n^{|m|}(\cos\phi), \tag{6.24}$$

for which the boundary condition $u(1,\theta,\phi) = f(\theta,\phi)$ becomes

$$f(\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{mn} e^{im\theta} P_n^{|m|}(\cos\phi). \tag{6.25}$$

Now, $\{e^{im\theta}\}_{m=-\infty}^{\infty}$ is a complete orthogonal set on $(-\pi,\pi)$, and since the substitution $s = \cos\phi$ gives

$$\int_{-1}^{1} F(s)\,ds = \int_0^{\pi} F(\cos\phi)\sin\phi\,d\phi$$

for any F, Theorems 6.1 and 6.7 show that for each m, $\{P_n^{|m|}(\cos\phi)\}_{n=|m|}^{\infty}$ is a complete orthogonal set on $(0,\pi)$ with respect to the weight function $\sin\phi$. It follows that the functions

$$Y_{mn}(\theta,\phi) = e^{im\theta} P_n^{|m|}(\phi) \qquad (n = 0,1,2,\ldots;\ |m| \le n),$$

considered as functions on the unit sphere S in $\mathbf{R}^3$, form an orthogonal basis of $L^2(S)$ with respect to the surface measure $d\sigma(\theta,\phi) = \sin\phi\,d\theta\,d\phi$. Moreover, the normalization constants can be read off from Theorems 6.1 and 6.7:

$$\|Y_{mn}\|^2 = 2\pi\frac{(n+|m|)!}{(n-|m|)!}\frac{2}{2n+1} = \frac{4\pi}{2n+1}\frac{(n+|m|)!}{(n-|m|)!}.$$

The functions Y_{mn} are called **spherical harmonics.**

The series (6.25) is just the expansion of f with respect to the basis of spherical harmonics, so the coefficients c_{mn} in (6.25) are given by

$$\begin{aligned} c_{mn} &= \frac{\langle f, Y_{mn}\rangle}{\|Y_{mn}\|^2} \\ &= \frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}\int_0^{\pi}\int_{-\pi}^{\pi} f(\theta,\phi)e^{-im\theta}P_n^{|m|}(\phi)\sin\phi\,d\theta\,d\phi. \end{aligned} \tag{6.26}$$

We have therefore proved the following result.

Theorem 6.8. *The solution of the Dirichlet problem (6.13) is the series (6.24) in which the coefficients c_{mn} are given by (6.26).*

This is not the end of the story, however. There are two additional important facts about the solution (6.24) that should be pointed out. The first, a significant feature that is obscured by the use of spherical coordinates, is that each term of the series in (6.24) is a homogeneous polynomial in the Cartesian coordinates (x,y,z). The second is that the infinite series in (6.24) can be re-expressed as an integral that is in some respects more useful; it is the 3-dimensional analogue of the Poisson integral formula that we presented in §4.4. We now discuss these two facts in the form of theorems.

Theorem 6.9. *For each m and n, the (m,n)th term in (6.24) is a homogeneous polynomial of degree n in the Cartesian coordinates (x, y, z).*

Proof: First consider the case $m \geq 0$. As in the proof of Theorem 6.7, we observe that $P_n^m(s) = (1-s^2)^{m/2} q_{n-m}(s)$ where q_{n-m} is a polynomial of degree $n-m$ that is even or odd according as $n-m$ is even or odd. Thus we can write $q_{n-m}(s) = \sum_{2j \leq n-m} a_j s^{n-m-2j}$, and hence

$$P_n^m(\cos\phi) = \sin^m \phi \sum_{2j \leq n-m} a_j \cos^{n-m-2j} \phi.$$

Therefore,

$$\begin{aligned} r^n e^{im\theta} P_n^m(\cos\phi) &= [re^{i\theta} \sin\phi]^m \sum_{2j \leq n-m} a_j r^{2j} (r\cos\phi)^{n-m-2j} \\ &= (x+iy)^m \sum_{2j \leq n-m} a_j (x^2+y^2+z^2)^j z^{n-m-2j}, \end{aligned}$$

which is a homogeneous polynomial of degree n. The same calculation shows that for $m < 0$,

$$r^n e^{im\theta} P_n^{|m|}(\cos\phi) = (x-iy)^{|m|} \sum_{2j \leq n-|m|} a_j (x^2+y^2+z^2)^j z^{n-|m|-2j}. \quad \blacksquare$$

Theorem 6.9 implies that the series (6.24), when rewritten in Cartesian coordinates, is just the Taylor series of the solution u about the origin. It also implies that the spherical harmonics Y_{mn} are the restrictions of homogeneous harmonic polynomials to the unit sphere. The theory of spherical harmonics can also be developed from the beginning from this point of view; see Folland [24], Stein-Weiss [49], or Walker [53].

Theorem 6.10. *If f is a continuous function on the unit sphere $|\mathbf{x}| = 1$, the solution of the Dirichlet problem*

$$\nabla^2 u(\mathbf{x}) = 0 \text{ for } |\mathbf{x}| < 1, \qquad u(\mathbf{x}) = f(\mathbf{x}) \text{ for } |\mathbf{x}| = 1$$

is given by

$$u(\mathbf{x}) = \frac{1}{4\pi} \iint_{|\mathbf{y}|=1} \frac{1-|\mathbf{x}|^2}{(1-2|\mathbf{x}|\cos\alpha + |\mathbf{x}|^2)^{3/2}} f(\mathbf{y})\, d\sigma(\mathbf{y}), \tag{6.27}$$

where α is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$ and σ is the surface measure on the unit sphere.

Proof: First suppose $\mathbf{x}$ is on the positive z-axis, so that the spherical coordinates of $\mathbf{x}$ are $(r, *, 0)$ (the θ coordinate is undefined along the z-axis). Since $P_n^{|m|}(1) = 0$ for $m \neq 0$ (obvious from the definition) and $P_n^0(1) = P_n(1) = 1$ (Corollary 6.1), by Theorem 6.8 we have

$$u(r, *, 0) = \sum_{n=0}^{\infty} c_n r^n, \qquad c_n = \frac{2n+1}{4\pi} \int_0^{\pi} \int_{-\pi}^{\pi} f(\theta, \phi) P_n(\cos\phi) \sin\phi \, d\theta \, d\phi.$$

In other words,

$$u(r, *, 0) = \frac{1}{4\pi} \int_0^{\pi} \int_{-\pi}^{\pi} \left[\sum_0^{\infty} (2n+1) P_n(\cos\phi) r^n \right] f(\theta, \phi) \sin\phi \, d\theta \, d\phi \tag{6.28}$$

But since

$$(2n+1) r^n = \left(2r \frac{d}{dr} + 1 \right) r^n,$$

by Theorem 6.5 we have

$$\sum_0^{\infty} (2n+1) P_n(\cos\phi) r^n = \left(2r \frac{d}{dr} + 1 \right) \frac{1}{(1 - 2r\cos\phi + r^2)^{1/2}}$$
$$= \frac{1 - r^2}{(1 - 2r\cos\phi + r^2)^{3/2}}.$$

If we substitute this into (6.28) and write $\mathbf{y}$ for the point whose spherical coordinates are $(1, \theta, \phi)$, we obtain (6.27) for the special case that $\mathbf{x}$ is on the positive z-axis. But (6.27) is expressed in a way that is independent of the particular Cartesian coordinate system used. Thus, given any vector $\mathbf{x}$, we can choose the positive z-axis to be in the same direction as $\mathbf{x}$, and the same reasoning then shows that (6.27) is valid at $\mathbf{x}$. ∎

A number of other boundary value problems involving the Laplacian in spherical coordinates can be solved by modifying the calculations leading up to Theorem 6.8. Some of these problems are examined in Exercises 3–6. We shall conclude this discussion by showing how to handle problems that lead to the equation $\nabla^2 u = -\mu^2 u$ rather than $\nabla^2 u = 0$.

Suppose, for example, that we wish to calculate the temperature $u(\mathbf{x}, t)$ in a solid ball of radius 1 given that the initial temperature is $f(\mathbf{x})$ and the surface of the ball is held at temperature zero:

$$u_t = k\nabla^2 u \text{ for } |\mathbf{x}| < 1, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}), \qquad u(\mathbf{x}, t) = 0 \text{ for } |\mathbf{x}| = 1. \tag{6.29}$$

We first separate out the t dependence by taking $u = X(\mathbf{x})T(t)$, which gives

$$\frac{T'}{kT} = \frac{\nabla^2 X}{X} = -\mu^2,$$

so that $T(t) = Ce^{-\mu^2 kt}$ and

$$\nabla^2 X = -\mu^2 X \quad \text{for } |\mathbf{x}| < 1, \qquad X(\mathbf{x}) = 0 \quad \text{for } |\mathbf{x}| = 1. \tag{6.30}$$

We now express $\mathbf{x}$ in spherical coordinates and proceed just as we did for the Dirichlet problem (6.13). The reader may verify without difficulty that if we take $X = R(r)\Theta(\theta)\Phi(\phi)$ in (6.30), we obtain exactly the same equations for Θ and Φ as we did before, so that $\Theta(\theta) = e^{im\theta}$ and $\Phi(\phi) = P_n^{|m|}(\cos\phi)$ where m and n are integers with $|m| \le n$. The only change is that instead of the Euler equation (6.23) for R, we obtain

$$r^2 R'' + 2rR' + [\mu^2 r^2 - n(n+1)]R = 0. \tag{6.31}$$

This is almost, but not quite, a Bessel equation. In fact, if we set

$$R(r) = r^{-1/2} g(r),$$

a simple calculation shows that (6.31) turns into

$$r^2 g''(r) + rg'(r) + [\mu^2 r^2 - (n + \tfrac{1}{2})^2]g(r) = 0.$$

As we observed at the beginning of Chapter 5, the change of variable $r \to r/\mu$ transforms this into Bessel's equation of order $n + \frac{1}{2}$. Therefore, the solutions of (6.31) that are finite at $r = 0$ are constant multiples of

$$R(r) = r^{-1/2} J_{n+(1/2)}(\mu r).$$

(Recall that the power series expansion of $J_{n+(1/2)}(\mu r)$ about $r = 0$ involves the powers $r^{n+(1/2)+2j}$ with $j \ge 0$, so the expansion of R involves the powers r^{n+2j}. These exponents are positive integers, so R is analytic at the origin.) The boundary condition in (6.30) becomes $R(1) = 0$, so μ must be one of the positive zeros of $J_{n+(1/2)}$. Denoting these zeros by $\mu_1^n, \mu_2^n, \ldots$, we arrive at the following family of solutions of the heat equation that vanish on the unit sphere:

$$u(r, \theta, \phi, t) = \sum_{l,m,n} c_{lmn} r^{-1/2} J_{n+(1/2)}(\mu_l^n r) e^{im\theta} P_n^{|m|}(\cos\phi) e^{-(\mu_l^n)^2 t}. \tag{6.32}$$

Since the spherical harmonics $e^{im\theta} P_n^{|m|}(\cos\phi)$ form a complete orthogonal set with respect to the surface measure $\sin\phi\, d\theta\, d\phi$ on the unit sphere, and the Bessel functions $J_{n+(1/2)}(\mu_l^n r)$ form (for each fixed n) a complete orthogonal set with respect to the measure $r\,dr$ on $(0, 1)$, it follows that the functions

$$F_{lmn}(r, \theta, \phi) = r^{-1/2} J_{n+(1/2)}(\mu_l^n r) e^{im\theta} P_n^{|m|}(\cos\phi)$$

in (6.32) form a complete orthogonal set with respect to the volume measure

$$dV(r, \theta, \phi) = r^2 \sin\phi\, dr\, d\theta\, d\phi$$

on the unit ball. The normalizations are given by Theorem 6.7 and Theorem 5.3:

$$\|F_{lmn}\|^2 = \frac{2\pi(n+|m|)!J_{n+(3/2)}(\mu_l^n)^2}{(2n+1)(n-|m|)!}$$

To solve problem (6.29) we merely have to expand f with respect to this basis and plug the resulting coefficients c_{kmn} into (6.32).

The Legendre polynomials P_n and associated Legendre functions P_n^m are special solutions of the Legendre equations (6.18) and (6.17). For other applications — for example, the solution of the Dirichlet problem in a conical region — it is important to study the general solutions of these equations and to allow the parameters λ and m to be arbitrary complex numbers. These solutions go under the general name of **Legendre functions**. Accounts of the theory of Legendre functions can be found in Erdélyi et al. [21], Hochstadt [30], and Lebedev [36].

There are several other coordinate systems in $\mathbf{R}^3$ in which the technique of separation of variables can be applied to the Laplace operator, including the so-called *spheroidal, toroidal,* and *bipolar* coordinates. Separation of variables in these coordinates can be used to solve, for example, the Dirichlet problem in the interior of an ellipsoid of revolution, the interior of a torus, or the region between two intersecting spheres; the solutions all involve the Legendre functions. For a detailed account of these matters, we refer the reader to Lebedev [36].

EXERCISES

1. Solve the following Dirichlet problem: $\nabla^2 u(r,\theta,\phi) = 0$ for $r < 1$, $u(1,\theta,\phi) = \cos\phi$ for $0 \le \phi \le \frac{1}{2}\pi$, $u(1,\theta,\phi) = 0$ for $\frac{1}{2}\pi \le \phi \le \pi$. (Hint: Exercise 8, §6.2.)
2. Solve the following Dirichlet problem: $\nabla^2 u(r,\theta,\phi) = 0$ for $r < 1$, $u(1,\theta,\phi) = \cos^2\theta \sin^2\phi$. Express the answer both in spherical coordinates and in Cartesian coordinates.
3. Solve the Dirichlet problem for the *exterior* of a sphere: $\nabla^2 u(r,\theta,\phi) = 0$ for $r > 1$, $u(1,\theta,\phi) = f(\theta,\phi)$, and $u(r,\theta,\phi) \to 0$ as $r \to \infty$.
4. Solve the following Dirichlet problem in the upper hemisphere $r < 1$, $\phi < \frac{1}{2}\pi$: $\nabla^2 u(r,\theta,\phi) = 0$ for $r < 1$ and $\phi < \frac{1}{2}\pi$, $u(1,\theta,\phi) = f(\phi)$ for $\phi < \frac{1}{2}\pi$, $u(r,\theta,\frac{1}{2}\pi) = 0$. (Hint: Theorem 6.4.) What is the answer, explicitly, when $f(\phi) \equiv 1$? (Use Exercise 7, §6.2.)
5. Suppose the base of the hemispherical solid $r < 1$, $\phi < \frac{1}{2}\pi$ is insulated while its spherical surface $r = 1$ is held at a steady temperature $f(\phi)$. Find the steady-state temperature in the solid. (Hint: Theorem 6.4.)
6. Solve the Dirichlet problem in a spherical shell $a < r < b$: $\nabla^2 u(r,\theta,\phi) = 0$ for $a < r < b$, $u(a,\theta,\phi) = f(\theta,\phi)$, $u(b,\theta,\phi) = g(\theta,\phi)$. (Hint: Do the cases $f = 0$ and $g = 0$ separately; then use superposition.)
7. Solve the wave equation for the vibrations in a spherical cavity when the boundary is held fixed: $u_{tt} = c^u\nabla^2 u$ for $r < 1$, $u(1,\theta,\phi,t) = 0$. (Find the general solution for arbitrary initial conditions.)

8. Solve the initial value problem $u_t = k\left[(1-x^2)u_x\right]_x$ for $-1 < x < 1$, $u(x,0) = f(x)$. (This could be a model for the diffusion of a liquid through a gel whose diffusivity at position x is proportional to $1 - x^2$.)

6.4 Hermite polynomials

The nth **Hermite polynomial** $H_n(x)$ is defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{6.33}$$

Simple calculations show that

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2,$$
$$H_3(x) = 8x^3 - 12x, \qquad H_4(x) = 16x^4 - 48x^2 + 12.$$

In general, we have

$$e^{-x^2} H_n(x) = (-1)^n \frac{d^n}{dx^n} e^{-x^2} = -\frac{d}{dx}[e^{-x^2} H_{n-1}(x)]$$
$$= e^{-x^2}[2xH_{n-1}(x) - H'_{n-1}(x)],$$

or

$$H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x), \tag{6.34}$$

which allows one to compute H_n by induction on n (see Exercise 1). In particular, it follows from this formula that the leading term of $H_n(x)$ is $2x$ times the leading term of $H_{n-1}(x)$, and hence that H_n is a polynomial of degree n whose leading term is $(2x)^n$. Moreover, since e^{-x^2} is an even function, H_n is even or odd according as n is even or odd.

We now investigate the orthogonality properties of the Hermite polynomials. We shall be working with the spaces $L^2(\mathbf{R})$ and $L^2_w(\mathbf{R})$ where

$$w(x) = e^{-x^2}.$$

The symbol w will always have this meaning throughout this section. For future reference we note that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} y^{-1/2} e^{-y}\, dy = \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Theorem 6.11. *The Hermite polynomials* $\{H_n\}_0^{\infty}$ *are orthogonal on* $\mathbf{R}$ *with respect to the weight function* $w(x) = e^{-x^2}$, *and*

$$\|H_n\|_w^2 = 2^n n! \sqrt{\pi}.$$

Proof: If f is any polynomial we have

$$\langle f, H_n \rangle_w = \int_{-\infty}^{\infty} f(x)H_n(x)e^{-x^2}dx = (-1)^n \int_{-\infty}^{\infty} f(x)\frac{d^n}{dx^n}e^{-x^2}dx$$
$$= \int_{-\infty}^{\infty} f^{(n)}(x)e^{-x^2}dx.$$

For the last equation we have integrated by parts n times; the boundary terms vanish because $P(x)e^{-x^2} \to 0$ as $x \to \pm\infty$ for any polynomial P. If f is a polynomial of degree less than n, and in particular if $f = H_m$ with $m < n$, then $f^{(n)} \equiv 0$ and hence $\langle f, H_n \rangle_w = 0$. This proves the orthogonality of the Hermite polynomials. On the other hand, if $f = H_n$ we have $f(x) = (2x)^n + \cdots$ and hence $f^{(n)} \equiv 2^n n!$, so

$$\|H_n\|_w^2 = 2^n n! \int_{-\infty}^{\infty} e^{-x^2}dx = 2^n n! \sqrt{\pi}. \quad \blacksquare$$

We next establish the completeness of the orthogonal set $\{H_n\}_0^\infty$ in $L_w^2(\mathbf{R})$. Actually we shall prove a slightly stronger statement, for use in the next section.

Theorem 6.12. *Suppose f is a function on $\mathbf{R}$ such that $|f(x)|e^{|tx|}e^{-x^2}$ is integrable on $\mathbf{R}$ for all $t \in \mathbf{R}$. If*

$$\int_{-\infty}^{\infty} f(x)P(x)e^{-x^2}dx = 0 \quad \textit{for all polynomials } P,$$

then $f = 0$ (almost everywhere).

Proof: Since $e^{itx} = \sum_0^\infty (itx)^n/n!$ and

$$\left|\sum_0^N \frac{(itx)^n}{n!}\right| \le \sum_0^\infty \frac{|tx|^n}{n!} = e^{|tx|} \quad \text{for all } N \ge 0,$$

the dominated convergence theorem (applied to the partial sums of the series) implies that

$$\int_{-\infty}^{\infty} e^{itx} f(x)e^{-x^2}dx = \sum_0^\infty \frac{(it)^n}{n!} \int_{-\infty}^{\infty} x^n f(x)e^{-x^2}dx.$$

The hypothesis on f implies that all the integrals on the right vanish. By the Fourier inversion theorem, which we shall prove in §7.2, it follows that $f(x)e^{-x^2} = 0$, and hence $f(x) = 0$, almost everywhere. ∎

Corollary 6.2. *The set $\{H_n\}_0^\infty$ is an orthogonal basis for $L_w^2(\mathbf{R})$.*

Proof: If $f \in L_w^2(\mathbf{R})$ and $\langle f, H_n \rangle_w = 0$ for all n, then $\langle f, P \rangle_w = 0$ for all polynomials P by Lemma 6.1, §6.1, and

$$\int_{-\infty}^{\infty} |f(x)|e^{|tx|}e^{-x^2}dx \le \left(\int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2}dx\right)^{1/2} \left(\int_{-\infty}^{\infty} e^{2|tx|}e^{-x^2}dx\right)^{1/2} < \infty$$

by the Cauchy-Schwarz inequality. It follows from Theorem 6.12 that $f = 0$ in $L_w^2(\mathbf{R})$. ∎

Our next step in investigating the Hermite functions is to derive their generating function.

Theorem 6.13. *For any $x \in \mathbf{R}$ and $z \in \mathbf{C}$ we have*

$$\sum_0^\infty H_n(x)\frac{z^n}{n!} = e^{2xz-z^2}. \tag{6.35}$$

Proof: This formula can be proved by the same method as Theorem 6.5 (see Exercise 7), but we shall adopt a different approach here. We begin by observing that if $u = x - z$ (where x is fixed) we have $d/du = -d/dz$, and hence

$$\frac{d^n}{dz^n}e^{-(x-z)^2}\bigg|_{z=0} = (-1)^n\frac{d^n}{du^n}e^{-u^2}\bigg|_{u=x} = e^{-u^2}H_n(u)\bigg|_{u=x} = e^{-x^2}H_n(x).$$

Therefore, by Taylor's formula,

$$e^{-(x-z)^2} = \sum_0^\infty e^{-x^2}H_n(x)\frac{z^n}{n!}.$$

Multiplying through by e^{x^2}, we obtain (6.35). ∎

Differentiation of (6.35) with respect to x yields

$$\sum_0^\infty H_n'(x)\frac{z^n}{n!} = 2ze^{-2xz-z^2} = 2\sum_0^\infty H_n(x)\frac{z^{n+1}}{n!} = 2\sum_1^\infty H_{n-1}(x)\frac{z^n}{(n-1)!},$$

where for the last equation we have made the substitution $n \to n-1$. If we equate coefficients of z^n on the left and right, we find that

$$H_0' = 0, \qquad H_n' = 2nH_{n-1} \quad \text{for } n > 0. \tag{6.36}$$

Combining (6.36) with (6.34) yields the recursion formula

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x) \tag{6.37}$$

and also the differential equation

$$H_n(x) = \frac{x}{n}H_n'(x) - \frac{1}{2n}H_n''(x),$$

or

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

This equation can be written in Sturm-Liouville form by multiplying through by e^{-x^2}:

$$\left[e^{-x^2}H_n'(x)\right]' + 2ne^{-x^2}H_n(x) = 0.$$

In short, the Hermite polynomials are the eigenfunctions for the singular Sturm-Liouville problem

$$\left[e^{-x^2}y'\right]' + \lambda e^{-x^2}y = 0, \qquad -\infty < x < \infty, \tag{6.38}$$

the only "boundary condition" being that the solutions are required to be in $L^2_w(\mathbf{R})$.

For many purposes it is preferable to replace the Hermite polynomials by the **Hermite functions** h_n defined by

$$h_n(x) = e^{-x^2/2}H_n(x).$$

See Figure 6.2. We summarize their properties in a theorem.

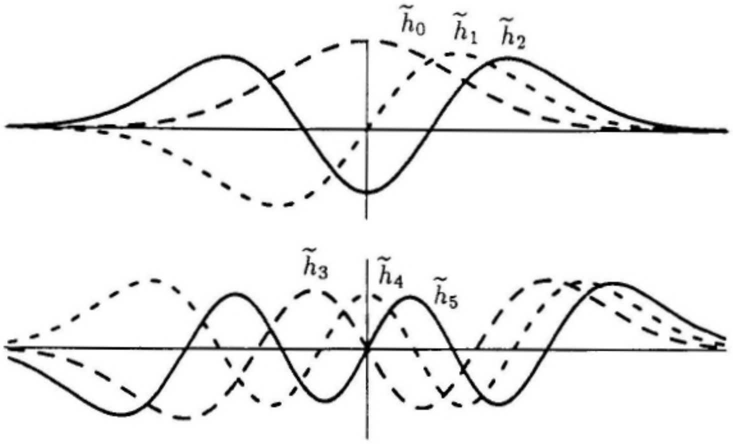


FIGURE 6.2. Graphs of some normalized Hermite functions $\widetilde{h}_n = h_n/\|h_n\| = h_n/\sqrt{2^n\pi^{1/2}n!}$ on the interval $-4 \le x \le 4$. Top: $\widetilde{h}_0$ (long dashes), $\widetilde{h}_1$ (short dashes), and $\widetilde{h}_2$ (solid). Bottom: $\widetilde{h}_3$ (long dashes), $\widetilde{h}_4$ (short dashes), and $\widetilde{h}_5$ (solid).

Theorem 6.14. *The Hermite functions* $\{h_n\}_0^\infty$ *are an orthogonal basis for* $L^2(\mathbf{R})$ *(with weight function* 1*). They satisfy*

$$xh_n(x) + h_n'(x) = 2nh_{n-1}(x), \tag{6.39}$$

$$xh_n(x) - h_n'(x) = h_{n+1}(x), \tag{6.40}$$

$$h_n''(x) - x^2h_n(x) + (2n+1)h_n(x) = 0. \tag{6.41}$$

Proof: The orthogonality of the h_n's follows from Theorem 6.11, since

$$\langle h_n, h_m\rangle = \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx = \langle H_n, H_m\rangle_w.$$

Similarly, their completeness follows from Theorem 6.12. If we write $H_n(x) = e^{x^2/2}h_n(x)$ in (6.36), we have

$$2ne^{x^2/2}h_{n-1}(x) = [e^{x^2/2}h_n(x)]' = e^{x^2/2}[xh_n(x) + h_n'(x)],$$

which is (6.39). In view of this result, (6.37) (with n replaced by $n+1$) becomes

$$h_{n+1}(x) = 2xh_n(x) - 2nh_{n-1}(x) = 2xh_n(x) - [xh_n(x) + h_n'(x)] = xh_n(x) - h_n'(x),$$

which is (6.40). Finally, if we combine (6.40) (with n replaced by $n-1$) and (6.39), we obtain

$$\begin{aligned} 2nh_n(x) &= 2n[xh_{n-1}(x) - h_{n-1}'(x)] = x[xh_n(x) + h_n'(x)] - [xh_n(x) + h_n'(x)]' \\ &= x^2h_n(x) - h_n''(x) - h_n(x), \end{aligned}$$

which is (6.41). ∎

Equation (6.41) shows that the Hermite functions are the L^2 eigenfunctions for the Sturm-Liouville equation

$$y'' - x^2 y + \lambda y = 0. \tag{6.42}$$

(6.42) and (6.38) are both referred to in the literature as the **Hermite equation.**

The Hermite equation (6.42) arises in the study of the classical boundary value problems in parabolic regions, through the use of **parabolic coordinates.** These are coordinates (s,t) in the plane related to the Cartesian coordinates (x,y) by

$$x = s^2 - t^2, \qquad y = 2st \qquad (-\infty < s < \infty,\ t \geq 0).$$

The curves $s = c$ and $t = c$ (c constant) are the parabolas $x = c^2 - (y/2c)^2$ and $x = (y/2c)^2 - c^2$ opening to the left or the right, with focus at the origin. See Figure 6.3.

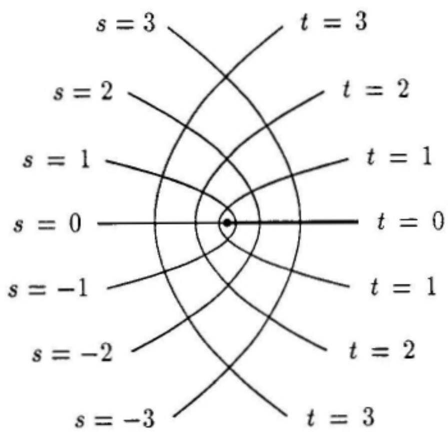


FIGURE 6.3. The parabolic coordinate system. This system is singular along the ray $t = 0$ (indicated by a heavy line), where the coordinates $(s, 0)$ and $(-s, 0)$ define the same point.

Let us consider the Laplace's equation in $\mathbf{R}^3$, in which we convert to parabolic coordinates in the xy-plane. A routine calculation with the chain rule shows that

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = \frac{1}{4(s^2+t^2)}(u_{ss} + u_{tt}) + u_{zz}.$$

As the reader may readily verify, if we substitute $u = S(s)T(t)Z(z)$ into the equation $\nabla^2 u = 0$ and separate out Z first, we obtain the ordinary differential equations

$$Z''(z) + \mu^2 Z(z) = 0,$$
$$S''(s) - 4\mu^2 s^2 S(s) + \lambda S(s) = 0,$$
$$T''(t) - 4\mu^2 t^2 T(x) - \lambda T(t) = 0.$$

Moreover, the substitutions $S(s) = f(\sqrt{2\mu}\, s)$ and $T(t) = g(i\sqrt{2\mu}\, t)$ convert the equations for S and T into the Hermite equation (6.42); more precisely, with $\sigma = \sqrt{2\mu}\, s$ and $\tau = i\sqrt{2\mu}\, t$ we have

$$f''(\sigma) - \sigma^2 f(\sigma) + \frac{\lambda}{2\mu} f(\sigma) = g''(\tau) - \tau^2 g(\tau) + \frac{\lambda}{2\mu} g(\tau) = 0.$$

Hence, by taking $\lambda = 2\mu(2n+1)$ we obtain the solutions

$$u = e^{\pm i\mu z} h_n(\sqrt{2\mu}\, s) h_n(i\sqrt{2\mu}\, t) = e^{\pm i\mu z} e^{2\mu(t^2 - s^2)} H_n(\sqrt{2\mu}\, s) H_n(i\sqrt{2\mu}\, t).$$

Of course, a complete analysis of the Laplacian in parabolic coordinates requires a study of all solutions of the Hermite equation (6.42) for arbitrary values of λ; these solutions are known as **parabolic cylinder functions**. This analysis is beyond the scope of this book, and we refer the reader to Erdélyi et al. [21] and Lebedev [36]; however, see Exercise 9.

The Hermite functions are also of importance in quantum mechanics, as they are the wave functions for the stationary states of the quantum harmonic oscillator. To be more precise, the wave functions for the stationary states of a quantum particle moving along a line in a potential $V(x)$ are the L^2 solutions of the equation

$$\frac{\hbar^2}{2m} u''(x) - V(x)u(x) + Eu(x) = 0,$$

where $\hbar$ is Planck's constant, m is the mass of the particle, and the eigenvalue E is the energy level. For a harmonic oscillator the potential is $V(x) = ax^2$ $(a > 0)$, so the substitutions $u(x) = f\big([2am/\hbar^2]^{1/4} x\big)$, $\xi = [2am/\hbar^2]^{1/4} x$ turn this equation into the Hermite equation

$$f''(\xi) - \xi^2 f(\xi) + \frac{\lambda}{\hbar}\sqrt{\frac{2m}{a}} = 0.$$

Thus the stationary wave functions are the Hermite functions $h_n\big([2am/\hbar^2]^{1/4} x\big)$, and the corresponding energy levels are $(2n+1)\hbar\sqrt{a/2m}$.

EXERCISES

1. Show by induction on n that

$$H_n(x) = n! \sum_{j \le n/2} \frac{(-1)^j (2x)^{n-2j}}{j!(n-2j)!}.$$

2. Find the general solution of the Hermite equation $y'' - 2xy' + \lambda y = 0$, where λ is an arbitrary complex number, by taking $y = \sum_0^\infty a_n x^n$ and solving for a_n in terms of a_0 and a_1. Show that the resulting series converge for all x.

3. Show that the Hermite equation in Exercise 2 has a polynomial solution of degree n precisely when $\lambda = 2n$, and that this solution is (a constant multiple of) H_n.
4. Expand the function $f(x) = x^{2m}$ (m a positive integer) in a series of Hermite polynomials. (Hint: Apply the formula used in the proof of Theorem 6.11.)
5. Expand the function $f(x) = e^{ax}$ in a series of Hermite polynomials. (Hint: Either proceed as in Exercise 4 or use Theorem 6.13.)
6. Let $f(x) = 1$ for $x > 0$, $f(x) = 0$ for $x < 0$. Expand f in a series of Hermite polynomials. (Hint: $e^{-x^2}H_n = -[e^{-x^2}H_{n-1}]'$. Use Exercise 1 to evaluate $H_n(0)$.)
7. Prove formula (6.35) by the method used to prove Theorem 6.5 — namely, plug definition (6.33) into the series $\sum H_n(x)z^n/n!$, apply the Cauchy integral formula for derivatives, sum the resulting geometric series, and finally apply the residue theorem.
8. Show that if $\phi_n(x) = h_n(ax)$ where $a > 0$, then $\{\phi_n\}_0^\infty$ is an orthogonal basis for $L^2(\mathbf{R})$, and that $\|\phi_n\|^2 = a^{-1}2^n n!\sqrt{\pi}$.
9. Let (s, t, z) denote parabolic coordinates in $\mathbf{R}^3$ as in the text. Consider the following Dirichlet problem in the parabolic slab $0 \le t \le 1$, $0 \le z \le 1$:

$$\nabla^2 u(s,t,z) = 0 \quad \text{for } t < 1,\ 0 < z < 1;$$
$$u(s,t,0) = u(s,t,1) = 0, \quad u(s,1,z) = f(s,z).$$

Assume that $\int_0^1 \int_{-\infty}^{\infty} |f(s,z)|^2 ds\, dz < \infty$ and find a solution in terms of Hermite functions.

6.5 Laguerre polynomials

Let α be a real number such that $\alpha > -1$. The nth **Laguerre polynomial** L_n^α corresponding to the parameter α is defined by

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x}). \tag{6.43}$$

(This formula makes perfectly good sense for any complex number α. However, it defines a polynomial of degree n only when α is not a negative integer, and these polynomials satisfy orthogonality relations only when $\alpha > -1$.) Some authors reserve the term *Laguerre polynomial* for the case $\alpha = 0$ and call the L_n^α's for $\alpha \neq 0$ *generalized Laguerre polynomials.*

By the product formula for nth derivatives, we have

$$\begin{aligned} L_n^\alpha(x) &= x^{-\alpha}e^x \sum_{k=0}^{n} \frac{1}{k!(n-k)!}\frac{d^k e^{-x}}{dx^k}\frac{d^{n-k}x^{\alpha+n}}{dx^{n-k}} \\ &= \sum_{k=0}^{n} \frac{(n+\alpha)(n-1+\alpha)\cdots(k+1-\alpha)}{k!(n-k)!}(-x)^k. \end{aligned} \tag{6.44}$$

Thus L_n^α is indeed a polynomial of degree n, and its leading term is $(-1)^n x^n/n!$. We shall now develop the basic properties of the polynomials L_n^α. The techniques will be very similar to those used for Legendre polynomials in §6.2, so we shall be brief.

Theorem 6.15. *The Laguerre polynomials $\{L_n^\alpha\}_{n=0}^\infty$ are a complete orthogonal set on $(0,\infty)$ with respect to the weight function*

$$w(x) = x^\alpha e^{-x},$$

and their norms are given by

$$\|L_n^\alpha\|_w^2 = \frac{\Gamma(n+\alpha+1)}{n!}.$$

Proof: If f is any polynomial, an n-fold integration by parts shows that

$$\begin{aligned}\int_0^\infty f(x)L_n^\alpha(x)x^\alpha e^{-x}\,dx &= \frac{1}{n!}\int_0^\infty f(x)\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x})\,dx \\ &= \frac{(-1)^n}{n!}\int_0^\infty f^{(n)}(x)x^{\alpha+n}e^{-x}\,dx.\end{aligned}$$

If f is of degree less than n, in particular if $f = L_m^\alpha$ with $m < n$, then $f^{(n)} \equiv 0$; this proves that $\langle L_n^\alpha, L_m^\alpha\rangle_w = 0$ for $n \neq m$. On the other hand, if $f = L_n^\alpha$ then $f^{(n)} \equiv (-1)^n$ by (6.44), so

$$\|L_n^\alpha\|_w^2 = \frac{1}{n!}\int_0^\infty x^{\alpha+n}e^{-x}\,dx = \frac{\Gamma(n+\alpha+1)}{n!}.$$

To prove completeness, we assume that $g \in L_w^2(0,\infty)$ satisfies $\langle g, L_n^\alpha\rangle = 0$ for all n and show that $g = 0$. To do this, we transfer the problem from $(0,\infty)$ to $(-\infty,\infty)$ by using the formula

$$\int_0^\infty F(x)\,dx = \int_0^\infty F(y^2)\,2y\,dy = \int_{-\infty}^\infty F(y^2)|y|\,dy, \tag{6.45}$$

valid for any integrable function F on $(0,\infty)$. To begin with, since every polynomial, and in particular every monomial x^n, is a linear combination of L_n^α's (by Lemma 6.1, §6.1), the conditions $\langle g, L_n^\alpha\rangle_w = 0$ together with (6.45) imply that

$$0 = \int_0^\infty g(x)x^n x^\alpha e^{-x}\,dx = \int_{-\infty}^\infty g(y^2)|y|^{2\alpha+1}y^{2n}e^{-y^2}\,dy$$

for all n. But also

$$0 = \int_{-\infty}^\infty g(y^2)|y|^{2\alpha+1}y^{2n+1}e^{-y^2}\,dy$$

for all n, simply because the integrand is an odd function. Therefore,

$$0 = \int_{-\infty}^{\infty} g(y^2)|y|^{2\alpha+1}P(y)e^{-y^2}\,dy$$

for every polynomial P. But then the function $f(y) = g(y^2)|y|^{2\alpha+1}$ satisfies the hypotheses of Theorem 6.12, for by the Cauchy-Schwarz inequality and (6.45),

$$\begin{aligned}
&\int_{-\infty}^{\infty} |g(y^2)||y|^{2\alpha+1}e^{|ty|}e^{-y^2}\,dy \\
&\quad\le \left(\int_{-\infty}^{\infty} |g(y^2)|^2|y|^{2\alpha+1}e^{-y^2}\,dy\right)^{1/2}\left(\int_{-\infty}^{\infty} |y|^{2\alpha+1}e^{2|ty|}e^{-y^2}\,dy\right)^{1/2} \\
&\quad= \|g\|_w\left(\int_{-\infty}^{\infty} |y|^{2\alpha+1}e^{2|ty|}e^{-y^2}\,dy\right)^{1/2} < \infty.
\end{aligned}$$

Therefore, by Theorem 6.12, $g = 0$. ▮

Remark. The assumption $\alpha > -1$ is necessary in Theorem 6.15. If $\alpha \le -1$ then the function $w(x) = x^\alpha e^{-x}$ is not integrable at the origin, so the integrals defining $\langle L_n^\alpha, L_k^\alpha\rangle_w$ and $\|L_n^\alpha\|_w^2$ all diverge.

We next show that the Laguerre polynomials satisfy the **Laguerre equation**

$$[x^{\alpha+1}e^{-x}y']' + nx^\alpha e^{-x}y = 0, \tag{6.46}$$

which can also be written in the form

$$xy'' + (\alpha+1-x)y' + ny = 0 \tag{6.47}$$

in view of the fact that

$$[x^{\alpha+1}e^{-x}y']' = x^{\alpha+1}e^{-x}y'' + (\alpha+1-x)x^\alpha e^{-x}y'. \tag{6.48}$$

Theorem 6.16. *The Laguerre polynomial L_n^α satisfies equation (6.46).*

Proof: Let $y_n = L_n^\alpha$. By (6.48),

$$[x^{\alpha+1}e^{-x}y_n']' = x^\alpha e^{-x}\left[-xy_n' + xy_n'' + (\alpha+1)y_n'\right].$$

The expression in square brackets on the right is a polynomial of degree n whose leading term is the leading term of $-xy_n'$. By (6.44), this is the same as the leading term of $-ny_n$, namely, $(-1)^{n-1}x^n/(n-1)!$. In other words,

$$[x^{\alpha+1}e^{-x}y_n']' = x^\alpha e^{-x}(-ny_n + P) \tag{6.49}$$

where P is a polynomial of degree less than n. By Lemma 6.1, §6.1, P must be a linear combination of the Laguerre polynomials $y_k = L_k^\alpha$ with $k < n$. We shall show that P is orthogonal to all these polynomials with respect to the weight $w(x) = x^\alpha e^{-x}$, from which it follows that $P = 0$ and hence that y_n satisfies (6.46).

Indeed, by (6.49),

$$\int_0^\infty P(x)y_k(x)x^\alpha e^{-x}\,dx = \int_0^\infty ny_n(x)y_k(x)x^\alpha e^{-x}\,dx + \int_0^\infty [x^{\alpha+1}e^{-x}y_n'(x)]'y_k(x)\,dx.$$

The first term on the right vanishes since $\langle y_n, y_k\rangle_w = 0$, and after two integrations by parts and another use of (6.48), the second term becomes

$$\int_0^\infty y_n(x)[x^{\alpha+1}e^{-x}y_k'(x)]'\,dx = \int_0^\infty y_n(x)Q(x)x^\alpha e^{-x}\,dx$$

where Q is a polynomial of degree k. But y_n is orthogonal to all such polynomials, so the integral vanishes. ∎

Theorem 6.16 implies that the Laguerre polynomials L_n^α are the eigenfunctions for a Sturm-Liouville problem on the interval $(0,\infty)$ associated to the differential equation

$$[x^{\alpha+1}e^{-x}y']' + \lambda x^\alpha e^{-x}y = 0. \tag{6.50}$$

This problem is singular both because $(0,\infty)$ is an infinite interval and because (6.50) has a regular singular point at $x = 0$. The boundary conditions for this problem are as follows. At infinity the only condition is that the solution should be square-integrable with respect to the weight $w(x) = x^\alpha e^{-x}$. On the other hand, from the theory of regular singular points, one knows that (6.50) has one solution that is analytic at $x = 0$ and another that is asymptotic to $x^{-\alpha}$ (or $\log x$ when $\alpha = 0$) as $x \to 0$. For $\alpha \geq 0$ the analytic solution is singled out by requiring it to remain finite as $x \to 0$, whereas for $-1 < \alpha < 0$ it is singled out by requiring that its first derivative remain finite as $x \to 0$.

We now derive the generating function for the Laguerre polynomials.

Theorem 6.17. *For $x > 0$ and $|z| < 1$,*

$$\sum_0^\infty L_n^\alpha(x)z^n = \frac{e^{-xz/(1-z)}}{(1-z)^{\alpha+1}} \tag{6.51}$$

Proof: The idea is the same as the proof of Theorem 6.5. Namely, if $x > 0$, let γ denote a circle in the right half-plane centered at x. We successively apply the defninion (6.43), the Cauchy integral formula for derivatives, the formula for the sum of a geometric series, the substitution $\sigma = (1-z)\zeta$ (which transforms γ into another circle γ'), and the Cauchy integral formula once again to obtain

$$\begin{aligned}
\sum_0^\infty L_n^\alpha(x)z^n &= \sum_0^\infty \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x})z^n \\
&= \frac{x^{-\alpha}e^x}{2\pi i}\sum_0^\infty z^n \int_\gamma \frac{\zeta^{\alpha+n}e^{-\zeta}}{(\zeta - x)^{n+1}}\,d\zeta \\
&= \frac{x^{-\alpha}e^x}{2\pi i}\int_\gamma \frac{\zeta^\alpha e^{-\zeta}}{\zeta - x}\sum_0^\infty \left(\frac{\zeta z}{\zeta - x}\right)^n d\zeta \\
&= \frac{x^{-\alpha}e^x}{2\pi i}\int_\gamma \frac{\zeta^\alpha e^{-\zeta}}{\zeta(1-z) - x}\,d\zeta \\
&= \frac{x^{-\alpha}e^x}{2\pi i(1-z)^{\alpha+1}}\int_{\gamma'} \frac{\sigma^\alpha e^{-\sigma/(1-z)}}{\sigma - x}\,d\sigma \\
&= \frac{x^{-\alpha}e^x}{(1-z)^{\alpha+1}}x^\alpha e^{-x/(1-z)}.
\end{aligned}$$

These formal calculations are valid for $|z|$ sufficiently small and prove (6.51) for such z; but then (6.51) is valid for $|z| < 1$ since the right side is analytic there. We leave the details of the justification to the reader. ∎

Perhaps the most striking application of Laguerre polynomials is in the quantum-mechanical analysis of the hydrogen atom. We shall sketch the ideas very briefly and refer the reader to Landau-Lifshitz [35] (whose notation for Laguerre polynomials, however, differs from ours) for a complete treatment.

Consider a system consisting of an electron and a proton. Since the proton is about 2,000 times more massive than the electron, we shall neglect its motion and consider it to be fixed at the origin. The electron is then moving in an electrostatic force field with potential $-\epsilon^2/r$ where ϵ is the charge of the proton and r is the distance from the origin. According to quantum mechanics, if the electron is in a stationary state at the energy level $E \in \mathbf{R}$, its wave function u is a function in $L^2(\mathbf{R}^3)$ satisfying the equation

$$\frac{\hbar^2}{2m}\nabla^2 u + \frac{\epsilon^2}{r}u + Eu = 0, \tag{6.52}$$

where $\hbar$ is Planck's constant and m is the mass of the electron. By an appropriate choice of units we may, and shall, assume that $\hbar = m = \epsilon = 1$.

We apply the method of separation of variables to solve (6.52), using spherical coordinates and taking $u = R(r)\Theta(\theta)\Phi(\phi)$. By the same calculations as in §6.3 we find that $\Theta(\theta) = e^{im\theta}$ and $\Phi(\phi) = P_n^{|m|}(\cos\phi)$ where m and n are integers with $n \geq |m|$, and R satisfies

$$r^2R'' + 2rR' + [2Er^2 + 2r - n(n+1)]R = 0. \tag{6.53}$$

We are primarily interested in the states where the energy level E is negative, that is, where the electron and proton are bound together in an atom. Assuming $E < 0$, then, we make the substitutions

$$\nu = (-2E)^{-1/2}, \qquad s = 2\nu^{-1}r, \qquad R(r) = S(2\nu^{-1}r) = S(s),$$

which (by a routine calculation) turn (6.53) into

$$s^2S'' + 2sS' + [\nu s - \tfrac{1}{4}s^2 - n(n+1)]S = 0. \tag{6.54}$$

Finally, we set $S = s^n e^{-s/2}\Sigma$ in (6.54), and after some more computation we obtain

$$s\Sigma'' + (2n+2-s)\Sigma' + (\nu - n - 1)\Sigma = 0.$$

This is the Laguerre equation (6.47) with $\alpha = 2n+1$ and n replaced by $\nu - n - 1$. The only solutions of this equation that lead to solutions $u = R\Theta\Phi$ of (6.52) that are in $L^2(\mathbf{R}^3)$ are the Laguerre polynomials. Hence ν must be an integer $\geq n+1$, and after reversing these substitutions we end up with the solution

$$R_{n\nu}(r) = (2\nu^{-1}r)^n e^{-r/\nu} L_{\nu-n-1}^{2n+1}(2\nu^{-1}r) \tag{6.55}$$

of (6.53). The eigenfunctions for the original problem (6.52) are

$$u_{mn\nu} = R_{n\nu}(r)e^{im\theta}P_n^{|m|}(\cos\phi) \qquad (|m| \leq n < \nu), \tag{6.56}$$

and the eigenvalue E of $u_{mn\nu}$ is $-\frac{1}{2}\nu^{-2}$.

We conclude with two important points concerning the physical interpretation of these results. First, when an electron jumps from one energy level $-\frac{1}{2}\nu^{-2}$ to a lower one $-\frac{1}{2}\mu^{-2}$, it emits a photon of frequency $(h/2)(\mu^{-2} - \nu^{-2})$. The fact that these frequencies are (up to a constant factor) differences of reciprocal squares of integers was known experimentally before the invention of quantum mechanics, and it provided one of the decisive early confirmations of the quantum theory.

Second, for each eigenvalue $-\frac{1}{2}\nu^{-2}$ with $\nu > 1$ there are several different eigenfunctions in the list (6.56). In fact, there are ν allowable values of n (namely, $0, \ldots, \nu - 1$), and for each such n there are $2n+1$ allowable values of m (namely, $-n, \ldots, n$). Hence for each ν there are

$$\sum_{n=0}^{\nu-1}(2n+1) = \nu^2$$

independent eigenfunctions. Actually, one must also take into account the spin of the electron, which has two eigenstates ("up"and "down"); this effectively doubles the number of independent eigenfunctions at each energy level. The collections of eigenfunctions at the various energy levels constitute the "electron shells" that form the basis for the periodic table of elements.

One final remark: The eigenfunctions (6.55) for the Sturm-Liouville problem (6.53) do *not* form a complete orthogonal set. Problem (6.53) has both a discrete and a continuous spectrum; this means that the expansion of a general function in terms of the eigenfunctions of (6.53) involves not only a sum over the eigenfunctions with eigenvalues $-\frac{1}{2}\nu^{-2}$ that we found above, but also an integral over a collection of (non-L^2) eigenfunctions corresponding to eigenvalues $E \geq 0$. Physically this means that an electron-proton system has not only bound states but also unbound states in which the electron has enough energy to escape from the electrostatic potential well. An analysis of the unbound states can be found in Landau-Lifshitz [35].

EXERCISES

1. Consider the Laguerre equation $xy'' + (\alpha + 1 - x)y' + \lambda y = 0$ where λ and α are arbitrary complex numbers.
   a. Assuming that α is not a negative integer, find a solution in the form $y = \sum_0^\infty a_n x^n$ with $a_0 = 1$, and show that this solution is a constant multiple of L_n^α when λ is a nonnegative integer n.
   b. Assuming that α is not a positive integer, find a solution in the form $y = \sum_0^\infty b_n x^{n-\alpha}$ with $b_0 = 1$.
2. By differentiating formula (6.51) with respect to z, show that

$$(1-z^2)\frac{\partial}{\partial z}\sum L_n^\alpha(x)z^n = \left[x + (1+\alpha)(z-1)\right]\sum L_n^\alpha(x)z^n,$$

   and hence derive the recursion formula

$$(n+1)L_{n+1}^\alpha(x) + (x - \alpha - 2n - 1)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0.$$

3. By differentiating formula (6.51) with respect to x as in Exercise 2, show that

$$(L_n^\alpha)'(x) - (L_{n-1}^\alpha)'(x) + L_{n-1}^\alpha(x) = 0.$$

4. Use formula (6.44) and Exercise 1, §6.4, to show that

$$L_n^{-1/2}(x) = \frac{(-1)^n}{2^{2n}n!}H_{2n}(\sqrt{x}), \qquad L_n^{1/2}(x) = \frac{(-1)^n}{2^{2n+1}n!}\frac{H_{2n+1}(\sqrt{x})}{\sqrt{x}}.$$

5. Expand the function $f(x) = x^\nu$ ($\nu \geq 0$) in a series of Laguerre polynomials. (Hint: To compute $\langle f, L_n^\alpha\rangle_w$, use formula (6.43) and integrate by parts n times.)
6. Expand the function $f(x) = e^{-bx}$ ($b > 0$) in a series of Laguerre polynomials. (Hint: Either proceed as in Exercise 5 or use Theorem 6.17.)

6.6 Other orthogonal bases

In this section we give a brief introduction to the other classical orthogonal sets of polynomials and to a few other orthonormal bases for L^2 spaces, not connected with differential equations, that have proved to be of importance.

Chebyshev* polynomials

The nth **Chebyshev polynomial** T_n is defined by the formula

$$T_n(\cos\theta) = \cos n\theta. \tag{6.57}$$

* The number of ways of transliterating the Russian name Chebyshev is almost infinite: Tchebyshev, Tchebichef, Tschebyschev, Čebyšev, etc.

Explicitly, we have

$$\cos n\theta = \operatorname{Re} e^{in\theta} = \operatorname{Re}(\cos\theta + i\sin\theta)^n = \operatorname{Re}\sum_{j=0}^{n} \frac{n!}{j!(n-j)!}(\cos\theta)^{n-j}(i\sin\theta)^j.$$

The real terms in the sum are those with j even, say $j = 2k$, and $(i\sin\theta)^{2k} = (\cos^2\theta - 1)^k$, so

$$\cos n\theta = \sum_{k\le n/2} \frac{n!}{(2k)!(n-2k)!}\cos^{n-2k}\theta\,(\cos^2\theta - 1)^k.$$

Therefore,

$$T_n(x) = \sum_{k\le n/2} \frac{n!}{(2k)!(n-2k)!}x^{n-2k}(x^2-1)^k.$$

Since $\{\cos n\theta\}_0^\infty$ is an orthogonal basis for $L^2(0,\pi)$, the substitution $\theta = \arccos x$ shows that $\{T_n\}_0^\infty$ is an orthogonal basis for $L_w^2(-1,1)$ where $w(x) = (1-x^2)^{-1/2}$. Indeed, if $m \neq n$,

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{(1-x^2)^{1/2}}\,dx = \int_0^\pi T_n(\cos\theta)T_m(\cos\theta)\,d\theta = \int_0^\pi \cos n\theta\cos m\theta\,d\theta = 0,$$

which gives the orthogonality. Likewise, if f is orthogonal to all T_n,

$$0 = \int_{-1}^{1} \frac{f(x)T_n(x)}{(1-x^2)^{1/2}}\,dx = \int_0^\pi f(\cos\theta)\cos n\theta\,d\theta,$$

whence $f = 0$; this gives the completeness. The same substitution shows that the differential equation $y'' + n^2y = 0$ for $\cos n\theta$ turns into the **Chebyshev equation**

$$(1-x^2)y'' - xy' - n^2y = 0, \qquad \text{or} \qquad \left[(1-x^2)^{1/2}y'\right]' + n^2(1-x^2)^{-1/2}y = 0,$$

satisfied by T_n.

The generating function for the Chebyshev polynomials is given by

$$1 + 2\sum_1^\infty T_n(x)z^n = \frac{1-z^2}{1-2xz+z^2}.$$

This formula is easily proved by substituting $x = \cos\theta$, writing

$$1 + 2\sum_1^\infty T_n(\cos\theta)z^n = 1 + 2\sum_1^\infty(\cos n\theta)z^n = \sum_{-\infty}^\infty e^{in\theta}z^n,$$

and summing the geometric series. We actually performed this calculation in §4.4, where this generating function (with $x = \cos\theta$ and $z = r$) turned out to be the Poisson kernel.

Chebyshev polynomials are of great importance in the theory of polynomial interpolation and approximation. We refer the reader to Rivlin [45] for a comprehensive account; see also Körner [34], §§43–45.

Jacobi polynomials

Let α and β be real numbers greater than -1. The nth **Jacobi polynomial** $P_n^{(\alpha,\beta)}$ associated to the parameters α and β is defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\Big[(1-x)^{\alpha+n}(1+x)^{\beta+n}\Big]. \tag{6.58}$$

When $\alpha = \beta = 0$, $P_n^{(\alpha,\beta)}$ is the Legendre polynomial P_n. The techniques we used in §6.2 to investigate the Legendre polynomials can be generalized to yield analogous results for the Jacobi polynomials:

(i) For each α and β, $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ is an orthogonal basis for $L_w^2(-1,1)$ where $w(x) = (1-x)^\alpha(1+x)^\beta$, and

$$\|P_n^{(\alpha,\beta)}\|_w = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.$$

(ii) $P_n^{(\alpha,\beta)}$ satisfies the **Jacobi equation**

$$(1-x^2)y'' + \Big[\beta-\alpha-(\alpha+\beta+2)x\Big]y' + n(n+\alpha+\beta+1)y = 0.$$

(iii) The generating function for the Jacobi polynomials $P_n^{(\alpha,\beta)}$ is

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)z^n = \frac{2^{\alpha+\beta}}{W(1-z+W)^\alpha(1+z+W)^\beta}, \qquad W = \sqrt{1-2xz+z^2}.$$

For more details, see Erdélyi et al. [21], Hochstadt [30], and Szegö [50].

We have observed that Legendre polynomials are the special case of Jacobi polynomials with $\alpha = \beta = 0$. Chebyshev polynomials are also essentially a special case of Jacobi polynomials, with $\alpha = \beta = -\frac{1}{2}$. Indeed, from the fact that $\{P_n^{(-1/2,-1/2)}\}$ and $\{T_n\}$ are both orthogonal bases for $L_w^2(-1,1)$ where $w(x) = (1-x^2)^{-1/2}$, or from the fact that the Jacobi differential equation reduces to the Chebyshev equation when $\alpha = \beta = -\frac{1}{2}$, it follows that T_n must be a constant multiple of $P_n^{(-1/2,-1/2)}$. In fact, it turns out that

$$T_n = \frac{2^{2n}(n!)^2}{(2n)!}P_n^{(-1/2,-1/2)}.$$

In the cases $\alpha = \beta > -\frac{1}{2}$, the Jacobi polynomials are sometimes given a different normalization and called **Gegenbauer polynomials** or **ultraspherical polynomials**. Precisely, the nth Gegenbauer polynomial C_n^λ associated to the parameter $\lambda > 0$ is defined by

$$C_n^\lambda(x) = \frac{\Gamma(2\lambda+n)\Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda)\Gamma(\lambda+n+\frac{1}{2})}P_n^{(\lambda-(1/2),\lambda-(1/2))}(x).$$

The reason for the new normalization is that the Gegenbauer polynomials have the simple generating function

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x) z^n = (1 - 2xz + z^2)^{-\lambda}.$$

The Jacobi polynomials $P_n^{((k-3)/2,\,(k-3)/2)}$, or equivalently the Gegenbauer polynomials $C_n^{(k-2)/2}$, play the same role in the theory of spherical harmonics in $\mathbf{R}^k$ as the Legendre polynomials do in $\mathbf{R}^3$; see Erdélyi et al. [21] and Stein-Weiss [49]. For some of the deeper properties and uses of Jacobi polynomials, see Askey [3].

Haar and Walsh functions

There are two interesting orthonormal bases for $L^2(0,1)$ consisting of step functions. The first one is the system of **Haar functions**

$$\{h_{(0)}\} \cup \{h_{jn} : j \geq 0,\ 0 \leq n < 2^j\}$$

constructed as follows:

$$h_{(0)}(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and for $j \geq 0$ and $0 \leq n < 2^j$,

$$h_{jn}(x) = \begin{cases} 2^{j/2} & \text{if } 2^{-j}n < x < 2^{-j}(n+\frac{1}{2}), \\ -2^{j/2} & \text{if } 2^{-j}(n+\frac{1}{2}) < x < 2^{-j}(n+1), \\ 0 & \text{otherwise.} \end{cases}$$

See Figure 6.4.

It is customary to parametrize the Haar functions by a single index m rather than the two indices j and n by defining

$$h_{(m)} = h_{jn} \text{ for } m = 2^j + n.$$

However, the use of two indices makes the geometry clearer; namely, j indicates the length of the interval on which h_{jn} is nonzero (to wit, 2^{-j}), whereas n indicates the position of that interval within $[0,1]$.

It is an easy exercise to see that the Haar functions are orthonormal. Indeed, the product $h_{jn}(x)h_{j'n'}(x)$ vanishes identically if $j = j'$ and $n \neq n'$, whereas if $j > j'$ it either vanishes identically or equals $\pm 2^{j'/2} h_{jn}(x)$. This is obvious if you think about the graphs of the h_{jn}'s for a minute; and it is equally obvious that $\int_0^1 h_{jn}(x)\,dx = 0$. Thus the h_{jn}'s are orthogonal to one another; similarly, $h_{jn}(x)h_{(0)}(x) = h_{jn}(x)$, so h_{jn} is orthogonal to $h_{(0)}$. Moreover,

$$\|h_{jn}\|^2 = \int_0^1 h_{jn}(x)^2\,dx = \int_{2^{-j}n}^{2^{-j}(n+1)} 2^j\,dx = 1.$$

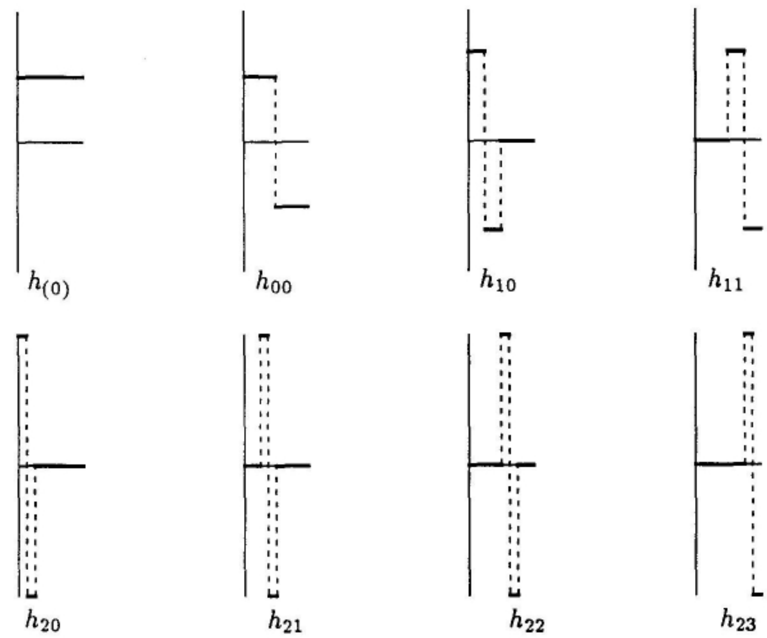


FIGURE 6.4. Graphs of the first eight Haar functions.

It is also easy to see that the Haar functions are complete in $L^2(0,1)$. The key observation is that the space of linear combinations of $h_{(0)}$ and the h_{jn}'s with $j < J$ equals the space of functions on $[0,1]$ that are constant on each interval $\left(2^{-J}k, 2^{-J}(k+1)\right)$ $(0 \le k < 2^J)$. (The former space is evidently contained in the latter one, and they both have dimension 2^J, so they coincide.) It follows that the set of all finite linear combinations of the Haar functions is the space of all step functions on $[0,1]$ whose discontinuities occur among the dyadic rational numbers $2^{-j}k$ $(j,k \ge 0)$, and this space is dense in $L^2(0,1)$.

In short, the Haar functions form an orthonormal basis for $L^2(0,1)$.

To construct our second orthonormal basis consisting of step functions, we begin with the **Rademacher functions** $r_n(x)$. For $n \ge 0$, one divides the interval $[0,1]$ into 2^n equal subintervals; $r_n(x)$ is the function which alternately takes the values $+1$ and -1 on these subintervals, beginning with $+1$ on the first subinterval. In other words, $r_n(x) = (-1)^{d_n(x)}$ where $d_n(x)$ is the nth digit in the binary decimal expansion of x. See Figure 6.5.

A **Walsh function** is a finite product of Rademacher functions. More precisely, if n is a nonnegative integer, let $b_k, \ldots, b_1$ be the digits in the binary decimal for n (i.e., $n = b_k \cdots b_1$ in base 2); then the nth Walsh function $w_n(x)$ is defined to be

$$w_n(x) = r_1(x)^{b_1} \cdots r_k(x)^{b_k}.$$

See Figure 6.6.

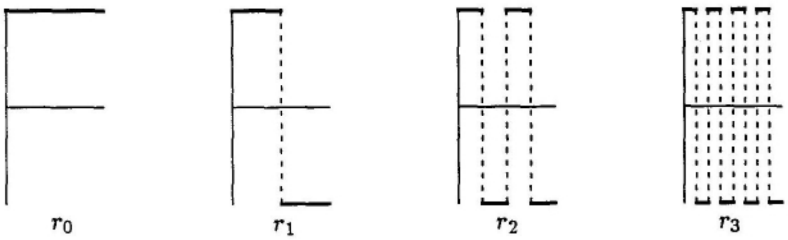


FIGURE 6.5. Graphs of the first four Rademacher functions.

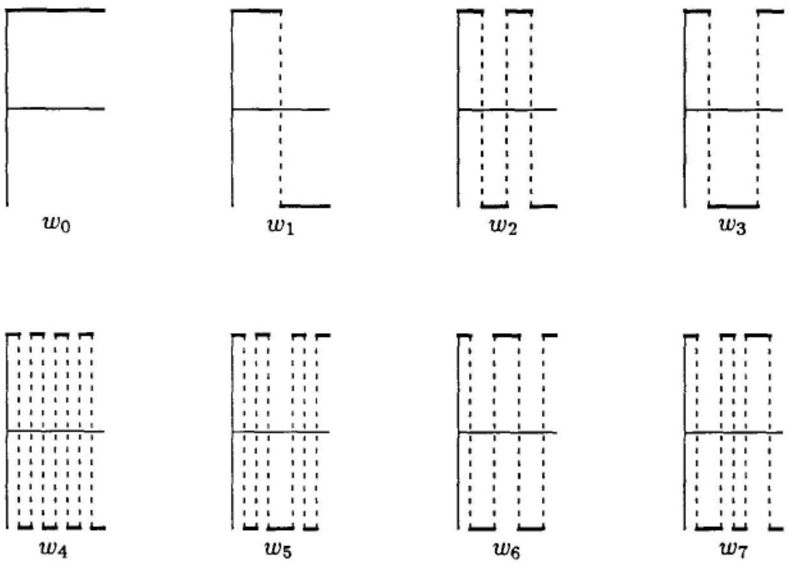


FIGURE 6.6. Graphs of the first eight Walsh functions.

The set $\{w_n\}_0^\infty$ of Walsh functions is an orthonormal basis for $L^2(0,1)$. Indeed, since the product of two Walsh functions is again a Walsh function, the orthogonality follows from the fact that $\int_0^1 w_n(x)\,dx = 0$ for $n > 0$; this, in turn, is true because the total length of the intervals on which $w_n(x) = 1$, and of the intervals on which $w_n(x) = -1$, is $\frac{1}{2}$. Also, $w_n(x)^2 \equiv 1$ (except at a finite number of points), so clearly $\|w_n\|^2 = 1$. The completeness follows by the same argument as for the Haar functions.

The property of the Haar functions that was emphasized by Haar in the 1910 paper where he introduced them is the fact that the expansion of any continuous function f on $[0,1]$ in a series of Haar functions converges uniformly to f — a feature that is conspicuously false for Fourier series and other orthogonal series arising from Sturm-Liouville problems. Walsh subsequently introduced his functions w_n in 1923 as an orthonormal set of step functions that qualitatively resemble the trigonometric functions more than the Haar functions, in that they live on the whole interval $[0, 1]$ rather than on small subintervals and become more and more oscillatory as n increases. Haar and Walsh functions have since been found to be interesting for various other theoretical reasons. They have also found a lot of practical applications due to their simplicity from the point of view of numerical calculations. In particular, the fact the the Walsh functions assume only the two values ± 1 makes them particularly handy to use with digital processing equipment. An account of the applications of Haar and Walsh functions in signal and image processing and related fields can be found in Beauchamp [4].

Wavelets

The Haar functions h_{jn} are generated from a single function by dilations and translations. Indeed, if

$$\chi(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$h_{jn}(x) = 2^{j/2}\chi(2^j x - n). \tag{6.59}$$

Since we were interested in functions on $[0, 1]$, we assumed that $0 \le n < 2^j$, but (6.59) makes sense for any integers j and n, and a modification of the arguments we gave above shows that $\{h_{jn}\}_{j,n=-\infty}^{\infty}$ is an orthonormal basis for $L^2(\mathbf{R})$. Like the Haar basis for $L^2(0, 1)$, this basis has both good and bad features. One of its main advantages is that it is "localized": The term $\langle f, h_{jn}\rangle h_{jn}$ in the expansion of a function f affects, and is affected by, the behavior of f *only* in the interval where $h_{jn} \neq 0$, and as $j \to +\infty$ this interval becomes smaller and smaller. Hence, in order to study the behavior of f in a small interval, one needs to look only at the terms in the series $\sum\langle f, h_{jn}\rangle h_{jn}$ such that h_{jn} "lives" on that interval. On the other hand, there is a disadvantage: When f is a smooth function the expansion $\sum\langle f, h_{jn}\rangle h_{jn}$ is only slowly convergent, and of course the partial sums are not smooth functions but step functions.

One of the exciting discoveries of recent years (1986–88, to be precise) is that this defect can be remedied while still preserving the localization property. In fact, we have the following result.

Theorem 6.18. *For any positive integer k there exist functions ψ of class $C^{(k)}$ on $\mathbf{R}$ that vanish outside a finite interval, such that the functions*

$$\psi_{jn}(x) = 2^{j/2}\psi(2^j x - n) \qquad (j, n = 0, \pm 1, \pm 2, \pm 3, \ldots)$$

constitute an orthonormal basis for $L^2(\mathbf{R})$.

The functions ψ_{jn} in this theorem are called **wavelets**; the basic function ψ is called the **mother wavelet**. The mother wavelets are *not* given by any simple formula but rather by a computationally effective recursive algorithm. Other constructions, involving spline (piecewise polynomial) functions or Fourier integrals, lead to variants of this theorem in which the mother wavelets $\psi(x)$ do not vanish outside a finite interval but do decay rapidly as $x \to \pm\infty$. (Theorem 6.18, as stated, is due to I. Daubechies; the variants just mentioned, which came a little earlier, are due to G. Battle, P. G. Lemarié, and Y. Meyer.)

Wavelet expansions share with Fourier series the property of being rapidly convergent when the function in question is smooth, but since wavelets (unlike trigonometric functions) are localized, one can use them to study *local* smoothness properties of functions. In fact, there is a close relationship between the smoothness properties of f near a point x_0 and the decay properties of the coefficients $\langle f, \psi_{jn}\rangle$ as $j \to +\infty$ for those j, n such that $\psi_{jn}(x_0) \neq 0$. (Of course this relationship holds only for properties involving only derivatives of order $\leq k$, k being the order of smoothness of the wavelets themselves.)

From a practical point of view, this has the following consequence. To be definite, let us consider a function $f \in L^2(\mathbf{R})$ that vanishes outside an interval $[-l, l]$. We can expand f in a Fourier series $\sum c_n e^{\pi inx/l}$ or a wavelet series $\sum\langle f, \psi_{jn}\rangle\psi_{jn}$. If f is everywhere smooth, these two representations of f are comparably efficient, that is, one has to take about the same number of terms in both cases to approximate f to a given accuracy. However, suppose f is smooth except for a small number of singularities such as jump discontinuities. The presence of even one singularity ruins the rapid convergence of the whole Fourier series, but the presence of a singularity at x_0 has little effect on the terms in the wavelet series except for the ones with $\psi_{jn}(x_0) \neq 0$. Hence, for functions with a small number of singularities the wavelet series is a much more efficient representation than the Fourier series. This makes wavelet series (and their higher-dimensional analogues) particularly useful in problems in signal and image processing having to do with edge detection and related phenomena.

The subject of wavelets and their applications (both in engineering and in pure mathematics) underwent an explosive development in the late 1980s. A more detailed discussion of these matters is beyond the scope of this book; we refer the reader to Daubechies [16], Mallat [38], and the articles by Daubechies and Meyer in [13].