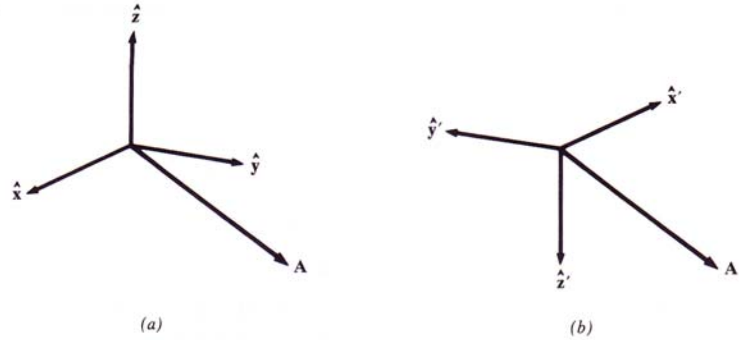


Figure 2 Coordinate inversion. (*a*) A vector **A** is shown together with a set of orthogonal coordinates. (*b*) A second set of coordinates is shown, related to the first set by inversion.

Polar and Axial Vectors. A second kind of coordinate transformation is shown in Figure 2. The operation that takes the unprimed into the primed coordinates is called inversion. If **A** is invariant under inversion, its components will be related by

$$A_x = -A_{x'} \qquad A_y = -A_{y'} \qquad A_z = -A_{z'} \tag{13}$$

A vector that transforms according to **(13)** is called a polar vector, the simplest example of which is a displacement.

It may be surprising that there exists a second class of vectors that do not transform in this way. The simplest example of this class is a rotation.

We show in Figure 3*a* a particle moving counterclockwise on a circle of radius R in the xy plane. It is convenient to characterize the motion of the particle by an angular frequency $\omega = v/R$.

We regard $\boldsymbol{\omega}$ as a vector along the axis about which the particle rotates. In order to relate a vector to a rotation we must use the "handedness" of the coordinate

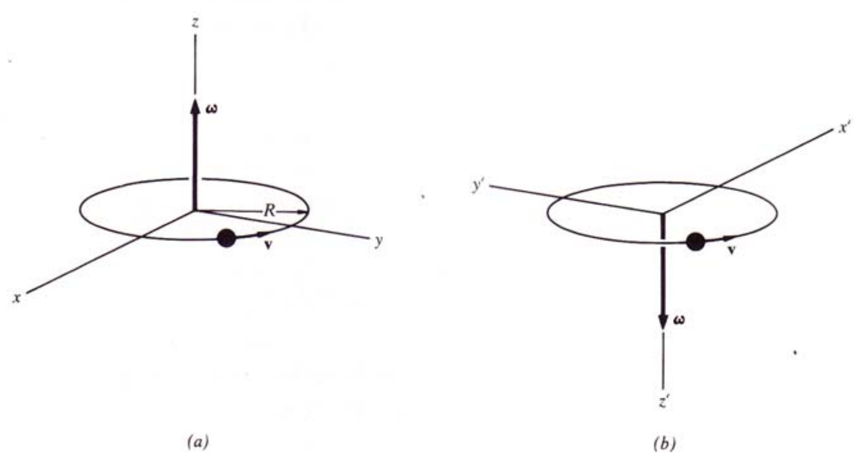


Figure 3 Axial vector. (*a*) The vector $\boldsymbol{\omega}$ represents positive (or counterclockwise) rotation in the x-y plane. (*b*) the vector $\boldsymbol{\omega}$ now represents positive (or *clockwise*) rotation in the x'-y' plane.

epresentation. Thus, in Figure 3*a* if we orient the fingers of our right hand in the ounterclockwise direction (from $\hat{\mathbf{x}}$ to $\hat{\mathbf{y}}$), our thumb points along $\hat{\mathbf{z}}$. For $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in yclic order, such a coordinate system is called righthanded. In order to generate $\hat{\mathbf{z}}'$ rom $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$ as shown in Figure 3*b*, we must use our left hand and such a coordinate ystem is called lefthanded.

Now, the direction that we take for $\boldsymbol{\omega}$ is dictated by the handedness of the oordinate representation. In Figure 3*a* we would take $\boldsymbol{\omega}$ in the positive $\hat{\mathbf{z}}$ direction:

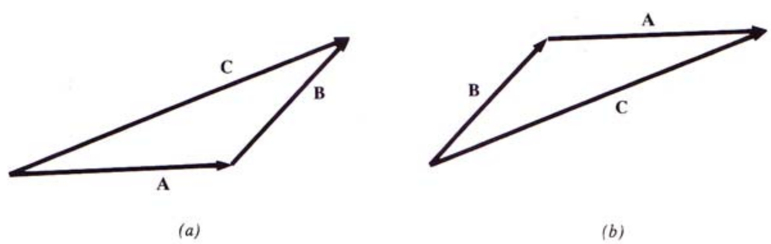
$$\boldsymbol{\omega} = \omega\hat{\mathbf{z}} \tag{14}$$

n Figure 3*b*, however, where the coordinate representation is lefthanded, we must take $\boldsymbol{\omega}$ in the positive $\hat{\mathbf{z}}'$ direction:

$$\boldsymbol{\omega} = \omega\hat{\mathbf{z}}' \tag{15}$$

For $\boldsymbol{\omega}$ in some general direction we have the transformation:

$$\omega_x = \omega_{x'} \qquad \omega_y = \omega_{y'} \qquad \omega_z = \omega_{z'} \tag{16}$$



igure 4 Law of triangles. (*a*) A pair of vectors **A** and **B** are added by forming a triangle s shown. (*b*) The resultant vector **C** is the same when **A** and **B** are interchanged, stablishing that vector addition is commutative.

 vector that transforms according to **(16)** under coordinate inversion is called an xial vector. The laws of physics may be expressed in terms of both polar and axial ectors although there are some restrictions imposed by the requirement that the laws hemselves be invariant under coordinate transformation.[2]

ddition of Vectors. Two vectors **A** and **B** are added by the geometrical construction hown in Figure 4*a*. The result of summing **A** and **B** is written as:

► $$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{17}$$

ote that the vector formed in Figure 4*b* by adding **A** onto **B** is also **C**. We write this onstruction as:

► $$\mathbf{C} = \mathbf{B} + \mathbf{A} \tag{18}$$

[2] A special situation is evidently presented by beta decay. See footnote 1.

uals the product of the area of its base and its altitude. For the parallelepiped the vectors **A**, **B**, and **C** shown in Fig. 1–11, the area of the base, which is the ea of the parallelogram formed by the vectors **B** and **C**, is given by the formula

$$\text{Area} = BC \sin \phi = |\mathbf{B} \times \mathbf{C}|. \tag{1–29}$$

The altitude h can be seen to be expressed by

$$h = \mathbf{A} \cdot \mathbf{n} = A \cos \theta,$$

here **n** is a unit vector perpendicular to the base. Thus we have for the volume the parallelepiped the formula

$$\text{Volume} = |\mathbf{B} \times \mathbf{C}| A \cos \theta = |\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}|. \tag{1–30}$$

here exists another useful triple product, the triple vector product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. is left as an exercise for the reader (Problem 1–5) to show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}. \tag{1–31}$$

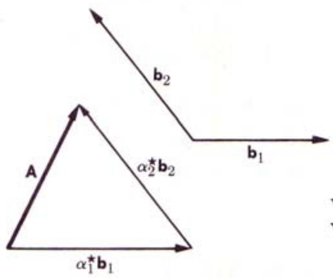


FIG. 1–12. Decomposition of a vector into the sum of two component vectors coplanar with the first.

-5 Nonorthogonal coordinate systems

It is of course not necessary and not always the most convenient choice to present a vector in terms of its components along three mutually perpendicular iit vectors. We therefore digress to discuss the representation of a vector **A** terms of a linear sum of three noncoplanar vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ (Fig. 1–12). If we set

$$\mathbf{A} = \alpha_1^\star \mathbf{b}_1 + \alpha_2^\star \mathbf{b}_2 + \alpha_3^\star \mathbf{b}_3, \tag{1–32}$$

here the $\alpha_i^\star$'s are constants, then by Eq. (1–17)

$$\begin{aligned} A_x &= \alpha_1^\star b_{1x} + \alpha_2^\star b_{2x} + \alpha_3^\star b_{3x}, \\ A_y &= \alpha_1^\star b_{1y} + \alpha_2^\star b_{2y} + \alpha_3^\star b_{3y}, \\ A_z &= \alpha_1^\star b_{1z} + \alpha_2^\star b_{2z} + \alpha_3^\star b_{3z}. \end{aligned} \tag{1–33}$$

quations (1–33) are three simultaneous linear equations which we may solve r the three constants $\alpha_1^\star$, $\alpha_2^\star$, and $\alpha_3^\star$. The solutions may be expressed in determinant notation. Thus we obtain

$$\alpha_1^\star = \frac{\begin{vmatrix} A_x & A_y & A_z \\ b_{2x} & b_{2y} & b_{2z} \\ b_{3x} & b_{3y} & b_{3z} \end{vmatrix}}{\begin{vmatrix} b_{1x} & b_{1y} & b_{1z} \\ b_{2x} & b_{2y} & b_{2z} \\ b_{3x} & b_{3y} & b_{3z} \end{vmatrix}} = \frac{\mathbf{A} \cdot \mathbf{b}_2 \times \mathbf{b}_3}{\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3}. \tag{1–34}$$

The last step follows from Eq. (1–27). Similarly, we obtain

$$\alpha_2^\star = \frac{\mathbf{A} \cdot \mathbf{b}_3 \times \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3} \qquad \text{and} \qquad \alpha_3^\star = \frac{\mathbf{A} \cdot \mathbf{b}_1 \times \mathbf{b}_2}{\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3}. \tag{1–35}$$

We realize, of course, that Eqs. (1–34) and (1–35) are uniquely soluble only if the scalar triple product of the three base vectors does not vanish,

$$\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3 \neq 0.$$

This condition is satisfied if the three base vectors are noncoplanar.

Equations (1–34) and (1–35) may be expressed more concisely in terms of the vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$, which are defined by the equations

$$\begin{aligned} \boldsymbol{b}_1 &= (\mathbf{b}_2 \times \mathbf{b}_3) \div (\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3), \\ \boldsymbol{b}_2 &= (\mathbf{b}_3 \times \mathbf{b}_1) \div (\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3), \\ \boldsymbol{b}_3 &= (\mathbf{b}_1 \times \mathbf{b}_2) \div (\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3), \end{aligned} \tag{1–36}$$

and which satisfy the relations

$$\boldsymbol{b}_1 \cdot \mathbf{b}_1 = \boldsymbol{b}_2 \cdot \mathbf{b}_2 = \boldsymbol{b}_3 \cdot \mathbf{b}_3 = 1$$

and (1–37)

$$\boldsymbol{b}_1 \cdot \mathbf{b}_2 = \boldsymbol{b}_1 \cdot \mathbf{b}_3 = \boldsymbol{b}_2 \cdot \mathbf{b}_1 = \boldsymbol{b}_2 \cdot \mathbf{b}_3 = \boldsymbol{b}_3 \cdot \mathbf{b}_1 = \boldsymbol{b}_3 \cdot \mathbf{b}_2 = 0.$$

The scalar products of the vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ and the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are concisely expressed by the equation

$$\boldsymbol{b}_i \cdot \mathbf{b}_j = \delta_{ij}, \qquad i, j = 1, 2, 3, \tag{1–38}$$

where δ_{ij} is the *Kronecker delta* having the value zero when $i \neq j$ and the value one when $i = j$, or

$$\delta_{ij} = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases} \tag{1–39}$$

In terms of the vectors $\boldsymbol{b}_i$,

$$\begin{aligned} \alpha_1^\star &= \mathbf{A} \cdot \boldsymbol{b}_1, \\ \alpha_2^\star &= \mathbf{A} \cdot \boldsymbol{b}_2, \\ \alpha_3^\star &= \mathbf{A} \cdot \boldsymbol{b}_3. \end{aligned} \tag{1–40}$$

The three vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ are referred to as the *inverse* or *reciprocal* ectors of the three vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, and the coordinate system formed by he vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ is referred to as the coordinate system reciprocal to he coordinate system formed by the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ (Fig. 1–13). We ote that a set of mutually orthogonal unit vectors is its own reciprocal set.

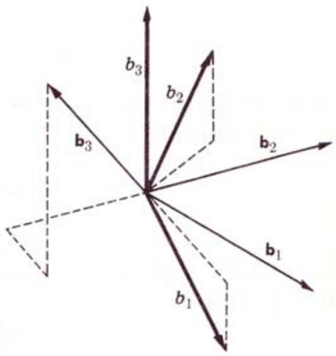


FIG. 1–13. Reciprocal sets of base vectors:

$\boldsymbol{b}_1 \perp \mathbf{b}_2, \mathbf{b}_3;\ \boldsymbol{b}_2 \perp \mathbf{b}_3, \mathbf{b}_1;\ \boldsymbol{b}_3 \perp \mathbf{b}_1, \mathbf{b}_2;$
$\mathbf{b}_1 \perp \boldsymbol{b}_2, \boldsymbol{b}_3;\ \mathbf{b}_2 \perp \boldsymbol{b}_3, \boldsymbol{b}_1;\ \mathbf{b}_3 \perp \boldsymbol{b}_1, \boldsymbol{b}_2.$

Through use of the definition of the inverse base vectors and Eq. (1–31), we btain the important relation

$$\begin{aligned} \boldsymbol{b}_1 \cdot \boldsymbol{b}_2 \times \boldsymbol{b}_3 &= \frac{(\mathbf{b}_2 \times \mathbf{b}_3) \cdot [(\mathbf{b}_3 \times \mathbf{b}_1) \times (\mathbf{b}_1 \times \mathbf{b}_2)]}{(\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3)^3} \\ &= \frac{(\mathbf{b}_2 \times \mathbf{b}_3) \cdot [(\mathbf{b}_3 \times \mathbf{b}_1 \cdot \mathbf{b}_2)\mathbf{b}_1]}{(\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3)^3} \\ &= \frac{1}{\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3} . \end{aligned} \tag{1–41}$$

Those familiar with the multiplication of determinants could have obtained this ame result through the product of the determinant

$$\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3 = \begin{vmatrix} b_{1x} & b_{1y} & b_{1z} \\ b_{2x} & b_{2y} & b_{2z} \\ b_{3x} & b_{3y} & b_{3z} \end{vmatrix}$$

ind the determinant

$$\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 \times \boldsymbol{b}_3 = \begin{vmatrix} b^\star_{1x} & b^\star_{1y} & b^\star_{1z} \\ b^\star_{2x} & b^\star_{2y} & b^\star_{2z} \\ b^\star_{3x} & b^\star_{3y} & b^\star_{3z} \end{vmatrix} = \begin{vmatrix} b^\star_{1x} & b^\star_{2x} & b^\star_{3x} \\ b^\star_{1y} & b^\star_{2y} & b^\star_{3y} \\ b^\star_{1z} & b^\star_{2z} & b^\star_{3z} \end{vmatrix} .$$

That is,

$$\begin{vmatrix} b_{1x} & b_{1y} & b_{1z} \\ b_{2x} & b_{2y} & b_{2z} \\ b_{3x} & b_{3y} & b_{3z} \end{vmatrix} \cdot \begin{vmatrix} b^\star_{1x} & b^\star_{2x} & b^\star_{3x} \\ b^\star_{1y} & b^\star_{2y} & b^\star_{3y} \\ b^\star_{1z} & b^\star_{2z} & b^\star_{3z} \end{vmatrix} = \begin{vmatrix} \mathbf{b}_1 \cdot \boldsymbol{b}_1 & \mathbf{b}_1 \cdot \boldsymbol{b}_2 & \mathbf{b}_1 \cdot \boldsymbol{b}_3 \\ \mathbf{b}_2 \cdot \boldsymbol{b}_1 & \mathbf{b}_2 \cdot \boldsymbol{b}_2 & \mathbf{b}_2 \cdot \boldsymbol{b}_3 \\ \mathbf{b}_3 \cdot \boldsymbol{b}_1 & \mathbf{b}_3 \cdot \boldsymbol{b}_2 & \mathbf{b}_3 \cdot \boldsymbol{b}_3 \end{vmatrix} = 1$$

▶ As an example, we consider the vector

$$\mathbf{A} = 5\mathbf{i} - 3\mathbf{j} + 8\mathbf{k},$$

which we seek to express as a linear sum of the vectors

$$\begin{aligned}\mathbf{b}_1 &= 3\mathbf{i} - 4\mathbf{j},\\ \mathbf{b}_2 &= 3\mathbf{j} + 4\mathbf{k},\\ \mathbf{b}_3 &= -\mathbf{i} + \mathbf{j} + 2\mathbf{k}.\end{aligned}$$

We first verify that the three vectors $\mathbf{b}_i$ are noncoplanar by evaluating their scalar triple product,

$$\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3 = \begin{vmatrix} 3 & -4 & 0 \\ 0 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} = 22.$$

Since the scalar triple product does not vanish, we can find a unique solution for the $\alpha_i^\star$. It is

$$\alpha_i^\star = \mathbf{A} \cdot \boldsymbol{b}_i.$$

Through use of Eq. (1–36), the reciprocal vectors $\boldsymbol{b}_i$ are found to be the vectors

$$\boldsymbol{b}_1 = \frac{1}{22}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} = \tfrac{1}{22}[2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}],$$

$$\boldsymbol{b}_2 = \frac{1}{22}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 3 & -4 & 0 \end{vmatrix} = \tfrac{1}{22}[8\mathbf{i} + 6\mathbf{j} + \mathbf{k}],$$

$$\boldsymbol{b}_3 = \frac{1}{22}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 0 \\ 0 & 3 & 4 \end{vmatrix} = \tfrac{1}{22}[-16\mathbf{i} - 12\mathbf{j} + 9\mathbf{k}],$$

yielding, by Eq. (1–40),

$$\begin{aligned}\alpha_1^\star &= \mathbf{A} \cdot \boldsymbol{b}_1 = \frac{5 \times 2 + (-3) \times (-4) + 8 \times 3}{22} = \frac{23}{11},\\ \alpha_2^\star &= \mathbf{A} \cdot \boldsymbol{b}_2 = \frac{5 \times 8 + (-3) \times (6) + 8 \times 1}{22} = \frac{15}{11},\\ \alpha_3^\star &= \mathbf{A} \cdot \boldsymbol{b}_3 = \frac{5 \times (-16) + (-3) \times (-12) + 8 \times 9}{22} = \frac{14}{11}.\end{aligned}$$

We have thus found that we can set

$$\begin{aligned}\mathbf{A} &= \alpha_1^\star \mathbf{b}_1 + \alpha_2^\star \mathbf{b}_2 + \alpha_3^\star \mathbf{b}_3\\ &= \tfrac{23}{11}\,\mathbf{b}_1 + \tfrac{15}{11}\mathbf{b}_2 + \tfrac{14}{11}\mathbf{b}_3.\end{aligned}$$

Checking, we find that this indeed yields

$$\begin{aligned}\mathbf{A} &= \tfrac{23}{11}(3\mathbf{i} - 4\mathbf{j}) + \tfrac{15}{11}(3\mathbf{j} + 4\mathbf{k}) + \tfrac{14}{11}(-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \\ &= 5\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}. \blacktriangleleft\end{aligned}$$

We could equally as well, of course, start our discussion with the reciprocal base vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$, and set

$$\mathbf{A} = \alpha_1 \boldsymbol{b}_1 + \alpha_2 \boldsymbol{b}_2 + \alpha_3 \boldsymbol{b}_3. \tag{1–42}$$

This would quite analogously lead us to the definition of the set of vectors which are reciprocal to the vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$. We find these to be the vectors (Problem 1–20)

$$\begin{aligned}\mathbf{b}_1 &= \frac{\boldsymbol{b}_2 \times \boldsymbol{b}_3}{\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 \times \boldsymbol{b}_3}, \\ \mathbf{b}_2 &= \frac{\boldsymbol{b}_3 \times \boldsymbol{b}_1}{\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 \times \boldsymbol{b}_3}, \\ \mathbf{b}_3 &= \frac{\boldsymbol{b}_1 \times \boldsymbol{b}_2}{\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 \times \boldsymbol{b}_3}.\end{aligned} \tag{1–43}$$

Analogously to the solution for the $\alpha_i^\star$, it follows that

$$\alpha_1 = \mathbf{A} \cdot \mathbf{b}_1, \qquad \alpha_2 = \mathbf{A} \cdot \mathbf{b}_2, \qquad \alpha_3 = \mathbf{A} \cdot \mathbf{b}_3. \tag{1–44}$$

▶Thus in the previous example we can also set

$$\mathbf{A} = \alpha_1 \boldsymbol{b}_1 + \alpha_2 \boldsymbol{b}_2 + \alpha_3 \boldsymbol{b}_3,$$

and obtain

$$\alpha_1 = \mathbf{A} \cdot \mathbf{b}_1 = 27, \qquad \alpha_2 = \mathbf{A} \cdot \mathbf{b}_2 = 23, \qquad \alpha_3 = \mathbf{A} \cdot \mathbf{b}_3 = 8.$$

That is,

$$\mathbf{A} = \tfrac{27}{22}(2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) + \tfrac{23}{22}(8\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + \tfrac{8}{22}(-16\mathbf{i} - 12\mathbf{j} + 9\mathbf{k}). \blacktriangleleft$$

We have thus arrived at the very important theorem that a three-dimensional vector is completely specified if its scalar products with three noncoplanar vectors is known.

It is interesting and extremely useful, as we shall see, to be able to compute the scalar and vector products of two vectors which are expressed in terms of their scalar products with three noncoplanar vectors. We find that the scalar product of two vectors **A** and **B** assumes its simplest form if we express one vector in terms of one set of noncoplanar base vectors and the other vector in terms of the reciprocal set of base vectors. If we thus set

$$\mathbf{A} = \alpha_1 \boldsymbol{b}_1 + \alpha_2 \boldsymbol{b}_2 + \alpha_3 \boldsymbol{b}_3 \tag{1–45}$$

and

$$\mathbf{B} = \beta_1^\star \mathbf{b}_1 + \beta_2^\star \mathbf{b}_2 + \beta_3^\star \mathbf{b}_3,$$

ve obtain through use of Eq. (1–37)

$$\mathbf{A} \cdot \mathbf{B} = \alpha_1\beta_1^\star + \alpha_2\beta_2^\star + \alpha_3\beta_3^\star, \tag{1–46}$$

r similarly,

$$\mathbf{B} \cdot \mathbf{A} = \beta_1\alpha_1^\star + \beta_2\alpha_2^\star + \beta_3\alpha_3^\star. \tag{1–47}$$

▶ Thus, for example, consider the scalar product of the vector

$$\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$$

vith the vector **A** of the previous example. We find that the vector **B** is expressi-le in the form

$$\mathbf{B} = 2\boldsymbol{b}_1 - 13\boldsymbol{b}_2 - 9\boldsymbol{b}_3,$$

vhere the $\boldsymbol{b}_i$ are the reciprocal vectors of the previous example. Hence by Eq. 1–47) the scalar product of **A** and **B** is found to be

$$\mathbf{B} \cdot \mathbf{A} = \tfrac{23}{11} \times 2 + \tfrac{15}{11}(-13) + \tfrac{14}{11}(-9) = -25.$$

This result is in agreement with the scalar product found by using the cartesian omponents of the vectors **A** and **B**,

$$\mathbf{A} \cdot \mathbf{B} = 5 \times 2 + (-3) \times (1) + 8 \times (-4) = -25. \blacktriangleleft$$

The vector product of two vectors **A** and **B** takes its simplest form when both ectors are expressed in terms of the same set of base vectors. Thus if we set

$$\mathbf{A} = \alpha_1^\star\mathbf{b}_1 + \alpha_2^\star\mathbf{b}_2 + \alpha_3^\star\mathbf{b}_3$$

nd

$$\mathbf{B} = \beta_1^\star\mathbf{b}_1 + \beta_2^\star\mathbf{b}_2 + \beta_3^\star\mathbf{b}_3,$$

hen we find that

$$\mathbf{A} \times \mathbf{B} = (\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3) \begin{vmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \\ \alpha_1^\star & \alpha_2^\star & \alpha_3^\star \\ \beta_1^\star & \beta_2^\star & \beta_3^\star \end{vmatrix} \cdot \tag{1–48}$$

imilarly, we find that

$$\mathbf{A} \times \mathbf{B} = (\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 \times \boldsymbol{b}_3) \begin{vmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} \cdot \tag{1–48a}$$

▶ Utilizing once more the vectors **A** and **B** of the previous example,

$$\mathbf{A} = 27\boldsymbol{b}_1 + 23\boldsymbol{b}_2 + 8\boldsymbol{b}_3$$

nd

$$\mathbf{B} = 2\boldsymbol{b}_1 - 13\boldsymbol{b}_2 - 9\boldsymbol{b}_3,$$

we find by Eq. (1–48a) that the vector product

$$\mathbf{A} \times \mathbf{B} = (b_1 \cdot b_2 \times b_3) \begin{vmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \\ 27 & 23 & 8 \\ 2 & -13 & -9 \end{vmatrix}$$
$$= \tfrac{1}{22}(-103\mathbf{b}_1 + 259\mathbf{b}_2 - 397\mathbf{b}_3)$$
$$= 4\mathbf{i} + 36\mathbf{j} + 11\mathbf{k}.$$

This result checks with the vector product found by utilizing the cartesian components of the vectors $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -3 & 8 \\ 2 & 1 & -4 \end{vmatrix} = 4\mathbf{i} + 36\mathbf{j} + 11\mathbf{k}. \blacktriangleleft$$

It should be apparent that we shall at times require some notation whereby we may recognize whether we expressed a vector in terms of the base vectors $\mathbf{b}_i$ or their reciprocal vectors b_i. To distinguish the two ways of expressing the vector $\mathbf{A}$ we shall whenever necessary let $\mathbf{A}$ represent the vector $\mathbf{A}$ expressed in terms of one set of base vectors and $\mathbf{A}^*$ the same vector $\mathbf{A}$ expressed in terms of the reciprocal base vectors. It does not matter how we choose the correspondence. The choice of the coordinate base vectors for the representation of $\mathbf{A}$ determines the representation of $\mathbf{A}^*$. Thus if

$$\mathbf{A}^* = \alpha_1^*\mathbf{b}_1 + \alpha_2^*\mathbf{b}_2 + \alpha_3^*\mathbf{b}_3, \tag{1–49}$$

then

$$\mathbf{A} = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3.$$

With this notation the scalar product of two vectors is most concisely represented by either

$$\mathbf{A} \cdot \mathbf{B}^* = \alpha_1\beta_1^* + \alpha_2\beta_2^* + \alpha_3\beta_3^* \tag{1–50}$$

or

$$\mathbf{A}^* \cdot \mathbf{B} = \alpha_1^*\beta_1 + \alpha_2^*\beta_2 + \alpha_3^*\beta_3.$$

That is,

$$\mathbf{A}^* \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B}^*.$$

Nonorthogonal base vectors are very important in physics. They are used extensively in problems dealing with the propagation of waves (electromagnetic, elastic, matter) in materials having a periodic structure as, for example, crystals.

An ideal crystal is a periodic structure, which is the same when viewed with respect to all points whose position is specified by

$$\mathbf{r} = \rho_1\mathbf{b}_1 + \rho_2\mathbf{b}_2 + \rho_3\mathbf{b}_3,$$